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ABSTRACT

This proceedings contains 75 research reports, 8 discussion groups, 32 oral reports, and 28 poster presentation entries from the 1996 Annual Meeting of the American Chapter of the International Group for the Psychology of Mathematics Education. A one-page synopsis is included for discussion groups, oral reports, and poster presentations. Topic areas include advanced mathematical thinking, algebraic thinking, assessment, cognitive modalities, curriculum reform, discourse, epistemology, functions and graphs, geometric thinking, probability and statistics, problem solving, rational number concepts, social and cultural factors, teacher beliefs and attitudes, teacher conceptions, teacher development, teacher understanding of student understanding, technology, and visualization. (AIM)

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ED 400 178

Proceedings of the Eighteenth Annual Meeting

North American Chapter
of the International Group
for the

Psychology of Mathematics Education

Volume 1: Plenary Papers, Discussion Groups, Research Papers, Short Oral Reports, and Poster Presentations

October 12-15, 1996
The Florida State University
Panama City, Florida U.S.A.

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Proceedings of the
Eighteenth Annual Meeting

North American Chapter
of the International Group
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Psychology of Mathematics Education

**Volume 1: Plenary Papers, Discussion Groups,
Research Papers, Oral Reports,
and Poster Presentations.**

PME-NA XVIII

October 12-15, 1996
The Florida State University
Panama City, Florida, U.S.A.

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Dierdre Watkins
Harry Biske

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History and Aims of the PME Group

PME came into existence at the Third International Congress on Mathematical Education (ICME 3) held in Karlsruhe, Germany, in 1976. It is affiliated with the International Commission for Mathematical Instruction.

The major goals of the International Group and of the North American Chapter (PME-NA) are:

1. To promote international contacts and the exchange of scientific information in the psychology of mathematics education.

2. To promote and stimulate interdisciplinary research in the aforesaid area with the cooperation of psychologists, mathematicians and mathematics teachers.

3. To further a deeper and better understanding of the psychological aspects of teaching and learning mathematics and the implications thereof.

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Preface

This program began with a meeting of interested volunteers on October 1995 in Columbus, OH during the PME-NA meeting. The results of the ideas discussed and suggestions made were taken to a meeting of the local program committee at The Florida State University where the themes of research cultures in North America, teaching and learning geometry, and social interactionism were selected. These themes became the focus of the three plenary sessions. Research cultures in North America are discussed in the three papers that were part of the plenary panel comprised of Fernando Hitt from Mexico, Carolyn Kieran from Canada, and Jeremy Kilpatrick from the United States. Learning and teaching geometry is addressed in the paper presented by Collete Laborde. Social interactionism as exemplified in learning environments is discussed by Heinrich Bauersfeld. The plenary panel, moderated by Norma Presmeg, ended with the audience discussing meta-questions raised through the three papers. No reaction papers were requested of the other two plenary papers in order to allow for participant reactions.

Included in the *Proceedings* are 75 research reports, eight discussion groups, 32 oral reports and 28 poster presentation entries. The one-page synopses of discussion groups, oral reports, and poster presentations are organized by topic along with the research reports following the pattern begun with the *Proceedings* of the 1994 PME-NA meeting. Proposers expressed first choice: 139 research reports, 20 oral presentations, 35 poster presentations, and eight discussion groups. Proposals for all categories were blind reviewed by three reviewers with expertise in the topic of submission. Cases of disagreement among reviewers were refereed by subcommittees of the Program Committee at The Florida State University. This process resulted in rejection or reassignment of about 43% of the research report proposals and about 37% overall.

Submissions for the *Proceedings* were made on disk, and produced by the ERIC/CSMEE staff. The format of the papers was adjusted to make them uniform. Papers are grouped in topic areas for the table of contents. An alphabetical list of addresses of authors is included in the appendix in Volume 2 with page numbers of their reports or synopses. In the case of multiple authors, submissions were made with presenting author(s) name(s) underlined.

The editors wish to express thanks to all those who submitted proposals, the reviewers, the 1996 Program Committee, the PME-NA Steering Committee for making the program an excellent contribution to ongoing research and discussions of psychology and mathematics education; Dean Jack Miller, College of Education, and the administration of the Department of Curriculum and Instruction and the Mathematics Education faculty at The Florida State University for their support; the graduate students for their endless work on the preparation of the conference and *Proceedings*; and the staff of the Gus Turnbull Center for Professional Development for the organization and registration of the conference.

Elizabeth Jakubowski
Dierdre Watkins
Harry Biske
October 1996

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PLENARY SESSIONS

RESEARCH CULTURE: RESEARCH ORIENTATION IN MATHEMATICS EDUCATION IN MEXICO

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Abstract: In the last fifteen years, research in mathematics education in Mexico has been oriented to: Curricular Development, Epistemological Analysis, Clinical Observation, Educational Experimentation (particularly with Teachers of Mathematics), Research on New Teaching Methods and Use of Technology, Classroom Research. Several concepts of Mathematics have been studied among teachers and students of different levels of education, in different environments. Trends of research are identified in this work. Some emerged naturally from institutional actions in Mexico to improve the learning and teaching of mathematics. Others showed the influence of research in other countries.

Introduction

The crisis produced in the 60's by the so called (in Mexico) *Modern Mathematics Reform*, encouraged the reflection about the difficulties to learn mathematics. By the seventies, systematic practice of research on mathematics education in Mexico, was being settled. Meanwhile, interdisciplinary groups of research were being formed in different countries to study phenomena linked with the learning of mathematics.

The Mexican Ministry of Education asked a group of mathematicians from the Advanced Studies and Research Center of the National Polytechnic Institute (Cinvestav-IPN) to elaborate the Mathematics textbooks for the elementary level (in Mexico, at the primary school grade 1-6, official textbooks are free and compulsory). This action promoted a change in the mathematics education in Mexico. The authors of those textbooks spotted the inefficacy of the *Modern Mathematics* stream in education and, despite their having teaching experience only at the university level, they accomplished the task at hand successfully (SEP Textbooks, 1972-75). As a consequence, for the primary school curriculum appeared brand new themes, v. gr., Probability and Statistics. However, the in-service teachers' lack of knowledge about several mathematics topics became a generalized disadvantage for successful teaching. That is, teachers were not prepared to achieve the aims of that change.

The problem of updating and training teachers led the group of researchers to create a Section of Educational Mathematics in the Cinvestav (1975, now Department), aiming to solve problems in the teaching and learning of mathematics. Since then, the Section has faced the task of overcoming those problems and promoting the development of similar groups along the Mexican Republic.

In the first stage, *the history of mathematics* and *curriculum development* were the main research subjects of the group. The product of this research is found in publications such as *Review of Mathematics and Teaching* (Revista Matemáticas y Enseñanza; Filloy, 1974, Alarcón, Gorostiza & Hernández, 1974, Rodríguez, 1974, Rivaud, 1976, 1996, Alarcón, 1979) and in the Official Mathematics Textbooks (1972-75).

Afterwards, the search for other ways to approach the problems of mathematics education could be approached, led the group to review and consider sources and research reports from different countries and trends. In particular, the link established with the IREM's at Bordeaux and Strasbourg turned out in the interest to focus on phenomena related to the learning of mathematics. Specifically, the notion of an epistemological obstacle from Bachelard (1971, 1975) and its interpretations and adaptations made by Brousseau (1976, 1983), oriented the Mexican researchers who pursued this line of investigation. On the one side, the history of mathematics provided a reference to analyze the kind of problems and reasonings and means to solve them, by considering their origin and evolution of concepts. On the other side, Brousseau's classification of epistemological, ontogenetic and didactic obstacles, were analyzed by the group and incorporated to their research. The works by Adda (1986), Glaeser (1978), Duval & Pluvinage (1977) and Pluvinage (1977), related to misunderstandings, heuristic, experimental education and data analysis, supplied a framework to which learner's performances, when being asked about mathematical concepts, could be analyzed.

The influence of U. S. researchers came from the work done by Taylor (1949) about curriculum, Bruner (1960) about education, Skinner (1972) with his Technology of Education, and by Bloom (1975) with the Educational Aims. The group had also access to the Soviet Literature from Krutetskii (1976) and the series of Soviet Studies in the Psychology of Learning and Teaching Mathematics (Kilpatrick & Wirszur, 1971). All this supplied a wide, but hard to conciliate, range of options, which were enriched by other works like those from Brunschvig (1912), Piaget (1960) and Piaget & García (1982).

Briefly, by the 80's we could identify the following trends of research:

- Curricular Development
- Exploratory Data Analysis
- Epistemological Analysis
- Clinical Observation

Curricular Development

Filloy (1981, pp. 239-240) identified the following as the main works in the curricular field:

- Production of materials in Dienes' tendency (Educational Research Department of Cinvestav, 1979, 1980).
- The Official Textbooks of Mathematics (1972-1975).
- High school books series, Alarcón et al. (1980, 1991, 1994, 1995) and Figueras et al. (1980).

One of the characteristics of this work was the rupture with the traditional way of "writing didactic material from the desk", and the beginning of a new focus with emphasis in classroom research.

The elementary education in Mexico goes from the age of 5 to 14 years and is compulsory. It is ruled by a unique curriculum all around the country. Unlike the primary level (6-11 years), for which there are official textbooks, as it was pointed out before; for the secondary school (12-14 years, corresponding more or less to the junior high school in the U. S.), there are a number of textbooks that can be used, no one being compulsory.

One of the directions followed by researchers interested in the secondary level included the analysis of official curriculum (Rojano, 1979, Recio, 1980) mainly by using theoretical elements supplied by the Taxonomies of Educational Aims (Bloom, 1972, 1975).

The curriculum of the pre-university level (high school in the U.S., grade 10-12) suffered a radical change, which brought out a different perspective for designed selection of the textbooks. Filloy (ibid.) quotes the didactic materials from López de Medrano (1972), which emphasized the idea of mathematizing real situations. These textbooks offered an alternative view which contrasted in character with the books previously used. However, the embodiment of the new materials and the current teaching lacked favorable conditions to implement the material. For example, there was not a teachers' preparation program to discuss the advantages and disadvantages of the textbooks proposed. As a consequence, the strength showed in the changes of perspective yielded little benefits and caused unsteadiness in the teachers.

The technological development brought out new aims for the mathematics curriculum. Students had to be educated in order to be served by and to profit from the new powerful tools. The fast proliferation of microcomputers and software produced mainly for purposes other than teaching, made that task compelling. Accordingly, another trend of research concerned the use of those tools for the teaching and learning of mathematical concepts. In particular, inquiries with respect to the use of commercial software already available are being carried out with pre-university an university students and teachers (Turriza, 1990; Riestra et al., 1990 and Chávez et al., 1993).

Nowadays, with the constructivist and problem solving orientation proposed for the pre-university level curriculum, educational authorities seem to take for granted success in accomplishing the new aims just by proposing, in their turn, syllabi and textbooks in correspondence with that orientation, overlooking the training and up-dating of teachers who have to put those elements into practice in their daily classroom tasks.

Exploratory Data Analysis (Quantitative Methods)

In the 70s and part of the 80s, quantitative methods prevailed in research all around the world, reaching also investigations in the educational field. In Mexico, the methods developed by Benzecri (1973) were adapted and incorporated to the research in the didactic of mathematics by Pluvinage (1977). For example, Techniques on Analysis of Correspondences (Analysis of Multidimensional Data) were applied to analyze a population in order to detect its sensibility to contradiction in mathematics (Hitt, 1978).

Later, the eruption of new research problems in other disciplines, and of ways to tackle them, gradually lessened the use of quantitative methods, as it was pointed out by

Kilpatrick (1995, pp. 4-6). Particularly, the studies carried out by Piaget et al. have strongly influenced the research in education, giving occasion to use qualitative methods to gather the information, aiming to explain rather than just to describe the educational phenomena, or to combine complementary views. The use of qualitative methods has characterized the research developed in Mexico in the 80's and 90's.

An instance of a research taking into account the two approaches is the one being carried out at the University of Morelos by Hernández & De Mata (in process), with 15-18 year old students. They investigate whether the stability of arithmetic-algebraic knowledge and problem solving in students can be better achieved by using arithmetic or algebraic strategies. The analysis has already yielded some results, such as: the progress in the understanding of some concepts is not significant after three years of instruction; even though 15 year old students solve the posed problems, they use arithmetic approaches more successfully in some problems than the algebraic ones used by the 18 year old students.

Clinical Observation

At the beginning of the 80's, some research questions focused on the emergence of particular ideas of mathematics in children and students, or on identifying misconceptions and their sources or in knowing students' strategies to solve problems. The need to get information on why subjects perform in particular ways when they face questions involving mathematical concepts, led researchers to make use of clinical observation. For example, Lema & Morfín (1981) undertook their work on the pertinence of Piaget & Inhelder's results (1951) about the idea of chance in Mexican children. They conducted individual interviews, in which the questioning concerning the experimental random situations took a shape of its own according to the successive answers given by the child.

Difficulties in reasonings that 12-13 year old children show in the transition from Arithmetic to Algebra, have been also investigated by means of clinical interviews (Rojano, 1985). Several problems have been detected and analyzed in this research project, such as the obstacles that students show to understand what a constant is, and the role that the unknown plays in an equation. The kinds of syntactic errors made when solving an equation have also been analyzed. In order to orientate didactic strategies to teach mathematics concepts at the primary and secondary levels of education, other researchers pointed to the convenience of knowing the way a semantic net of constructs about rational numbers is developed in pupils, or how arithmetic-algebraic skills evolve, or how verbal arithmetic-algebraic problems are solved (Figueras, 1987, 1988, 1996; Rubio, 1994). Complementary research on this trend has been conducted by Valdemoros (1993, 1994, 1995) and Dávila (1992), related to the concept of fractions in elementary school children.

A psychogenetic research also has been carried out in arithmetic and algebra. For example, in elementary school children, Bollás & Sánchez (1994) have employed a psychogenetic approach to study the concept of numbers, and Balbuena et al. (1993) in concepts of division. On the notion of variation, there is a work of Hoyos (1994), and in the study of Rosas (1995) algebra comprehension and rational numbers were investigated. Regarding problems with the concept of negative numbers, there are the works by Gallardo & Rojano (1990) and Gallardo (1993, 1994), that combined clinical observation and an historical approach to study the notion of negative numbers. On generalization procedures it is the Ursini's work (1990, 1991). Regarding algebraic errors, there is the work of Ortega (1990) & Avila et al. (1990), and Guzman's work (1993) about the algebraic use of language in problems modelation. On illiterate adults research it is Valiente (1995) about operator algorithms.

Research in pre-university students' understanding of fundamental ideas of stochastics has combined qualitative and quantitative approaches. After reviewing these studies, it seems doubtful that questions about the probability of intersection of events can be interpreted as conditional probability, as it was reported by some authors. The influence that the use of graphic representations in teaching (the idea of conditional probability) has in students' performances is highlighted (Ojeda, 1990, 1994, 1996). Alarcón's (1996) has made an analysis of students' probabilistic reasoning in parallel to their idea of proportional reasoning.

At the elementary level of education, in more complex settings, the data gathered from researcher's observation concerns the development of, for example, lessons of probability from which it is possible to identify the way teachers get didactic strategies underway, and to study the corresponding pupils' performances (Limón, 1995 ; Peláez, in process).

Santos (1993, 1994, 1995) has developed a line of research in mathematical problem solving, following very close Schoenfeld's theory, also there is a complement with Greeno's theory of transfer of situated knowledge. In other study, Santos (1996) presents examples illustrating mathematics problems in context which contribute to different kinds of knowledge in the students (inert, naive, ritual). This knowledge, in almost all the cases, obstructs the transfer of mathematical contents.

Results from the clinical observation in the research carried out by Filloy et al. (Kieran & Filloy, 1989 ; Filloy, 1990, 1991 ; Filloy & Hoyos, 1993 ; Hoyos, 1993 ; Filloy & Rubio, 1993 ; Filloy, 1996), have developed *alocal theoretical model* and *mathematical sign systems*. Filloy proposes three components for any theoretical model in research in mathematics education: *Models of teaching, Models of cognitive processes* and *Models of formal competence*.

Classroom Research

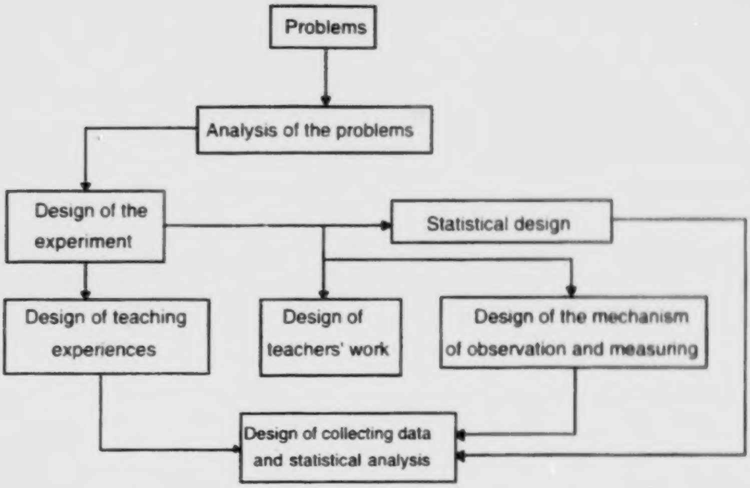
This kind of research is concerned with processes developed in natural environments in classrooms. For instance, Garnica (in process) is developing a research about the processes of communication in the classroom of the primary school, where there is the interplay of three fields: the universal pragmatic, the communicative action, and the communication. This research analyzes the process of learning not separated from the process of teaching.

Epistemological Analysis

The epistemological analysis of mathematical concepts has proved to be a convenient approach to understanding learning problems in mathematics. The detection of epistemological obstacles was one of the first tasks the group of researchers in Mexico faced (1974-77). This trend of research developed in a natural way because the mathematicians were the first interested in the problems of teaching and learning mathematics. The analysis of the history of mathematics yielded elements to be considered for the design of lessons, containing attractive and interesting didactic materials (see for example, Review of Teaching and Mathematics : Rivaud, 1976 ; Filloy, 1974-79). This was followed by the incorporation in Mexican research of Bachelard's work on epistemology (1971, 1975). Then, from 1978 to 1996, the group was concerned with the detection of epistemological obstacles by means of historical critical analysis. On this trend, research was carried out on concepts related to geometry, precalculus, calculus and analysis. For example, Antolín (1981, 1991) and Farfán (1986) did an historical analysis on the concept of function from Bernoulli (1718) to Fourier (1822). From a historical analysis from the Greeks to XVIII century, Bromberg & Moreno (1990) showed how non-euclidean geometry was discovered. Cantoral (1986) and Cordero (1986) presented different stages in the conceptual evolution of calculus. The concept of number and variation, and the concept evolution of infinity, were analyzed by Moreno (1991), Moreno and Waldegg (1987, 1991, 1995), Rigo (1994) and Waldegg (1992).

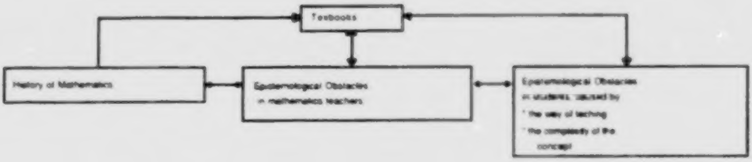
Experimentation and Research with Mathematics Teachers: Detection of Epistemological Obstacles

At the end of the 70s, research was carried out focusing on mathematics teachers. Filloy et al. (1977-79) investigated whether teachers of mathematics are good predictors of pupils' performances in simple tasks and bad predictors in complex ones. This research was organized as follows (Filloy et al., 1977-78, p. 5):

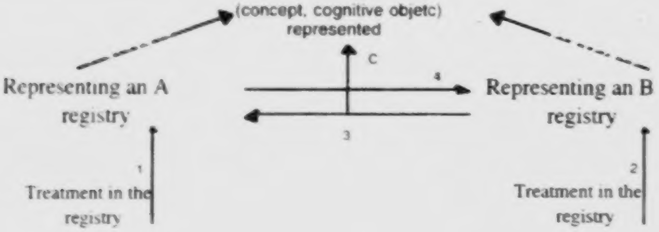

Problems
Analysis of the problems
Design of the experiment
Statistical design
Design of teaching experiences
Design of teachers' work
Design of the mechanism of observation and measuring
Design of collecting data and statistical analysis

A similar approach has been followed by Filloy et al. (1979) in their research with fifth year primary children's knowledge of decimal numbers; whereas a quantitative account of the collected information was reported by Hitt (1980). These studies involved a statistical design in their analysis.

By 1984, the Ministry of Education started the National Program for In-Service Mathematics Teachers Training and Up-Dating (PNFAPM), which was designed and implemented during one decade by the Department of Educational Mathematics together with 16 universities from other states around the country. This produced the beginning of a trend of research concerning the detection of obstacles in understanding mathematics concepts in mathematics teachers. These studies had the structure shown below, in which the statistical design does not play an important role:


Textbooks
History of Mathematics
Epistemological Obstacles in mathematics teachers
Epistemological Obstacles in students, caused by
* the way of teaching
* the complexity of the concept

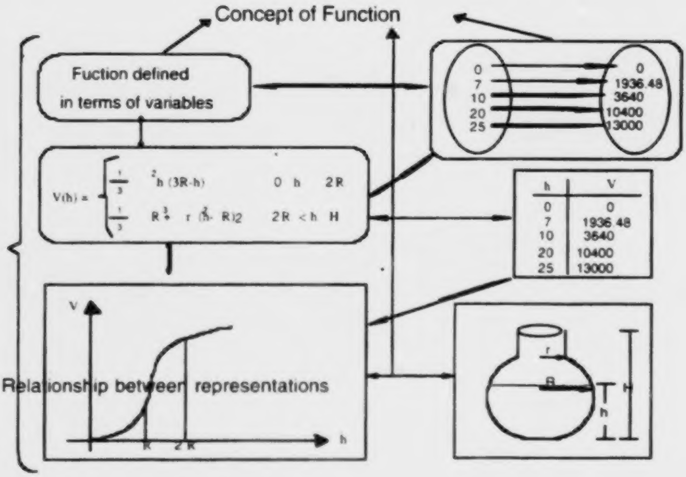
Several studies on historic analysis complemented by the detection of epistemological obstacles in students in areas such as precalculus, calculus and analysis were carried out (Arreguín, 1989 ; Cantoral, 1990 ; Cordero, 1993 ; Resendiz & Cordero, 1993 ; Sacristán, 1990 ; Zubieta & Moreno, 1996). Another area of interest was the detection of epistemological obstacles in teachers of mathematics (Cambray, 1993 ; Cordero, 1993 ; Farfán & Hitt, 1990 ; Farfán, 1993 ; Hitt, 1989, 1994, 1996). In particular, Hitt (1989, 1994, 1996) studied teachers' understanding of the concept of function through the way they articulated some of its different representations. In the research, two kinds of obstacles have been identified: epistemological and didactic. An example of an epistemological obstacle is the idea that high school teachers have of the concept of function as similar to the intuitive idea of function-continuity emerged at first in mathematics (Euler, 1748, pp. 1-3). Didactic obstacles (due to the way of teaching) have also been studied, such as: the different definitions of function presented by the authors of the textbooks may cause different obstacles to understand the concept, so it is convenient to look for the one which is preferable to use at this level of education. The inquiry was framed according to Duval's conception of representation (1993, p. 51; 1995, p. 67):



Following this view, *knowledge relative to a concept is stable in a subject if he/she is able to articulate the different representations of the concept without showing any contradiction* (Hitt, 1995, p. 64). The figure on the following page shows the need for interaction between the different external representations of the concept of function (ibidem, p. 64).

Another trend of research, with primary teachers, concerns the algorithms in the solving of problems (Block et al., 1990).

Sánchez (1996), by analyzing obstacles that mathematics teachers have to understand the concept of independence in probability, found that their misunderstandings are rooted in internal representations derived from their past experiences.



Research on New Methods of Teaching and the Using of Technology

The research being carried out in Mexico in this line intends to analyze processes of learning by using new methods of teaching through audiovisuals, calculators and microcomputers.

In the 80s, the Section of Educational Mathematics started a project on the production and research of audiovisual resources for the teaching of mathematics. The history of mathematics was the main support to the scripts according to which sequences of slides synchronized with audiotapes would convey information about the origin of mathematical concepts or theories. Examples of this work include: Archimeds' Method (Cantoral & Ojeda, 1985), Science and Art (Gallardo, 1981); Genesis of a Theory (Probability) (Ojeda, 1985), and Undefined Expansion of Universe (Ursini, 1983). Given the high cost of production and the anachronistic equipment (audioviewer), this project did not continue.

At the same time, another project started with the design of software for the teaching and learning of mathematics. As an example, lessons as a support for a course of analytical geometry were produced (Cuevas, Mejía & Riestra, 1985). The software produced allowed students to explore algebraic representation of a conic given its graphical representation. If the student gave a different expression, the software would plot his/her proposal. This line of research has continued producing software that assists in the learning of mathematics. The software provides students with significant didactic elements which are linked to research in mathematics education (for instance, on the spright line, Cuevas (1994) and Cortés (1995); and on the evaluation of analytic geometry concepts, Mejía (in process)).

The British influence on Mexican research is present in the work on computational microworlds (Hoyles & Noss, 1989). Software about polygonal numbers has been developed to work in paper, pencil and microcomputer environments (Hitt, 1994, 1996; Hitt & Monsoy, 1996).

Rojano et al. (1996) show that it is feasible to modify the present mathematics practice in classrooms by proposing activities that include computational environments (spreadsheets).

The use of graphic calculators has been studied in the learning of the concept of function (Balderas, 1995; Martínez, 1996); specifically, work has been carried out on trigonometric functions and learning models (Santillán, 1996; Wenzelburger, 1993).

Epistemological analysis and the use of new technologies are brought into play in a study carried out in the learning of differential equations (Hernández, 1995). In this study there is a clear French influence that is, the framework used in the analysis of the information includes concepts such as didactic transposition (Chevallard, 1985), interplay settings (Duady, 1986), numerical, algebraic and graphic settings (Artigue, 1989).

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MATHEMATICS EDUCATION RESEARCH IN CANADA: A PORTRAIT OF THE ECLECTIC

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In his chapter, titled "International Perspectives on Research in Mathematics Education" in the *Handbook of Research on Mathematics Teaching and Learning*, Bishop (1992) stated:

> It is my contention that our research field has grown from a situation where, even if it was once possible to talk of national perspectives, given the recency of organized mathematics education research in most countries it is now not only more difficult, but also, in my view, relatively unproductive to continue to do that. As a result of an increase in international meetings and conferences, international cooperative research projects, and books, reports, and journals coming from different countries, there is an obvious increase in the range of international influences to which researchers are subjecting themselves. As the number of people engaged in research in mathematics education has increased, and as they have made their ideas more public and shareable, so there has been a growing mutual international influencing of ideas, methods, practices, and expectations. (p. 710)

I tend to agree with the view expressed by Bishop. But in addition to Canadians' being open to international influences, Canada's size, location, historical temperament, and federal structure are such that a national perspective on mathematics education research would be quite unfeasible. Nevertheless, this panel has been given the charge to examine possible differences in the research cultures of our three North American countries--United States, Mexico, and Canada. To this end, the panelists have agreed to not only summarize recent research developments in their respective countries, but also look at factors outside the content and methodology of the research that is conducted, that is, see if there are some differences in how research is organized, conducted, and managed in our countries. I will try to communicate a sense of Canadian perspectives on some of these issues by touching on the nature of research funding in Canada, the areas that are being researched, and some of the theoretical orientations that are held, and then conclude with a few remarks on the eclectic nature of mathematics education research in Canada.

Research Funding in Canada

Ever since Canada was constituted in 1867, education has been under the control of the provinces; thus there is no national mathematics curriculum (for a history of school mathematics in Canada, see several chapters in the 32nd NCTM Yearbook, edited by Jones & Coxford, 1970). As well, there is no national voice speaking about mathematics education to governments and the public. But the major funding agency for mathematics

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education research is federal: the Social Sciences and Humanities Research Council of Canada (SSHRC). Only one province has set up its own research funding agency, that of Québec: Fonds pour la Formation de Chercheurs et l'Aide à la Recherche (FCAR). Thus, education researchers in Québec have access to two Canadian sources of research funds--the implications of which will be seen later.

SSHRC was set up in 1977; prior to that date it was the Canada Council that was the prime source of federal funding for Canadian research and development in the social sciences and humanities. (According to the first *Annual Report* of SSHRC in 1978-1979, there was one mathematics education project that was funded, that of D. Sawada from Alberta, titled "Information matching across sense modalities as related to mathematics learning.") SSHRC's aim has been to "support such discipline-based research as in the judgment of scholars will best advance knowledge; and encourage research on subjects which the Council, in consultation with the academic community, considers to be of national importance" (SSHRC, 1988-1989). Thus, SSHRC funds both targeted and independent research; but mathematics education researchers have in general availed themselves more often of the independent research program rather than that of targeted research. In awarding its research grants, the objectives of SSHRC were recast in 1988 in such a way that greater emphasis was placed on the previous research achievements of an applicant and the awarding of grants for broad, ongoing, well-planned programs of research rather than specific, short-term projects. The shift to program funding was intended to give researchers the kind of flexibility and encouragement that would allow them to build up a body of scholarship on a subject or on related topics. In balance, new scholars were to be judged more on their research proposal than on their earlier achievements. The adjudication of research proposals, from both experienced and new researchers, is assisted by a large pool of qualified reviewers from several countries.

Grants from the SSHRC tend now to be of three years' duration and never include indirect funds for the institution of the researcher and only rarely provide release time for the researchers. Thus, Canadian mathematics education university researchers carry out their research in addition to assuming their normal teaching load, which varies from one institution to another, but usually comprises two to three 45-hour classes during each of two semesters. In the most recent SSHRC *Annual Report* that is currently available, that of 1994-1995, of the six new mathematics education projects that were funded, the total amounts awarded for the 3-year-period for each project varied from 47 000$ to 160 000$. (The success rate at SSHRC for Standard Research Grants overall in 1994-95 was 46%--788/1732.) Figure 1 provides the province-by-province breakdown of SSHRC-funded projects in mathematics education for the 10-year period 1985-1995 (the first year in which a project was funded is the year that is used in Figure 1). In order to give a fairly current picture, I have elected to not go back farther than 10 years, even though mathematics education research has been taking place in Canada for many years prior to 1985 (Kieran, in preparation).

	BC	ALTA	SASK	MAN	ONT	QUÉ	NB	NS	PEI	NFLD
1994-95		1				5				
1993-94	1	1			1	4				
1992-93					1	2				
1991-92	1	1				1				
1990-91	1	1				4				
1989-90			1			6				
1988-89	1	1			2	2				
1987-88					2	5				
1986-87	1					3				
1985-86		1			1	2				

Figure 1. Canadian research in mathematics education funded by SSHRC (the first year in which a project was funded is the year used) during the 10-year period 1985-1995, province by province. Source: SSHRC *Annual Reports*.

It is clear from Figure 1 that more SSHRC-funded research in mathematics education takes place in the province of Québec than elsewhere. One of the major reasons for this is that research is very strongly supported by the government of Québec by means of its own funding agency, the FCAR, set up in 1984 to replace the similar FCAC that had been in existence since 1970. [During the mid-1960s, the educational system in Québec was drastically overhauled in what has been termed the Quiet Revolution; shortly afterwards the Québec government soon put money aside for educational research.] The FCAR aims "to promote and aid financially research which takes place in postsecondary institutions, . . . and to promote and aid financially the training of future researchers." Teams of at least two established researchers are judged on the scientific quality of the members of the team, their cohesion, the quality of the proposed researched activities, and the quality of the research milieu in terms of the training of researchers. The last criterion is just as important as each of the former three; proposals that do not include provision for the training of students as researchers have little chance of being funded. The SSHRC has recently begun to move in this direction as well. Both SSHRC and FCAR have grant programs for young scholars/new researchers aimed at "aiding new researchers who are beginning their careers to become autonomous researchers and to become competitive in both the national and international arenas" (translated from Guidelines of FCAR, 1996-1997).

The impact of the availability of FCAR research funds to Quebec researchers cannot be overemphasized. I commented above that the 1978-1979 *Annual Report* of SSHRC listed one new project that was funded in mathematics education; in contrast, the 1978-1979 listing of Québec FCAC recipients funded in mathematics education (FCAC, 1978-1979) mentioned seven new projects, whose principal investigators were N. Bednarz, M. Bélanger, V. Byers, C. DeFlandre, J. Hillel, D. Lunkenbein, and E. Quintin (a listing of the project titles, translated into English, can be found in Appendix 2). Provincial government support of this magnitude has thus not only encouraged the development of educational research in Québec but has also resulted in the creation of such an active research community in the province that it should not be surprising that the number of Québecers currently receiving grants from the federal agency SSHRC is so high.

In addition to the funds available from the SSHRC and FCAR for research in mathematics education, other Canadian sources--but to a much lesser degree--include universities, provincial Ministries of Education in conjunction with local school boards, and industry. It has been rare, however, to find joint venture grants involving industry and mathematics education researchers (the few exceptions have included projects with a strong technological component and with clear business applications). The meager funds available from Ministries of Education and school boards are usually targeted to specific problems of a quite local nature. As far as university funding is concerned, this source tends to be rather modest in nature, designed to encourage new research (or researchers) or to serve as a temporary bridge between more significant grants. Canadian mathematics education researchers are also eligible for funding by a handful of non-Canadian research agencies or as co-researchers on international grants, but such are beyond the scope of this panel. Because of the nature of these alternate Canadian sources of research funds, many mathematics education researchers in general rely on SSHRC and, if Québecers, FCAR as well. Some also manage to do research with no funding at all, but these cases tend to be rather the exception.

Areas Being Researched

A thematic synthesis of the project titles recently funded by SSHRC gives a good first approximation of the research areas being studied by Canadian mathematics education researchers. Such a synthesis also provides a fairly accurate picture of the research funded by FCAR since Québec mathematics education researchers have tended over the last 15 years or so to be funded as well by SSHRC for related components of their research programs. Figure 2 presents a thematic overview of the projects (and their principal investigators) funded by SSHRC during the same 10-year period as was used in Figure 1, that is, from 1985 to 1995. (The categories used in Figure 2 have a certain overlap; nevertheless, each project was fitted into only one category. For example, if computers figured prominently in the project title, it was placed in that category, even though it might also have dealt with learning. Individuals who were funded more than once in the same thematic area are indicated by a superscript showing the number of times. Appendix 1 contains the full project titles and names of co-investigators for each of the projects referred to in Figure 2.)

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1. Student learning in various mathematical domains (Allaire, Bednarz, Bergeron, Coron, Gaulin^2, Giroux, Hanrahan, Nantais, Owens^3, Pallascio, Randhawa, Sierpinska, Wheeler^3)
2. Use of computers in mathematics learning (Bélanger, Boileau^2, Brine [T], Hillel^2, C. Kieran, T. Kieren [T], J. Lajoie [T], S. Lajoie)
3. Problem solving in arithmetic, algebra, or geometry (Bednarz^2, Biron, C. Janvier [T], Poirier, René de Cotret)
4. Preservice and in-service teachers' mathematical knowledge and their use of conceptual models, etc. (Geddis, Kuendiger, Sigurdson, Zazkis)
5. Women in mathematics (Maclean [T], Mura [T], P. Rogers^2 [T])
6. Development of models of learning/understanding (Herscovics, T. Kieren^2)
7. Assessment (Dassa, W. Rogers)
8. Use of history of mathematics (Kleiner^2)
9. Socio-cultural conditions of mathematical learning (Zack)
10. Literary approach (Sawada)
11. Mathematical texts (Mura [T])
12. Cross-cultural studies (Pallascio [T])

Figure 2. Thematic overview of projects (and their principal investigators) funded by SSHRC during the period 1985-1995. [T] indicates targeted research. Source: SSHRC *Annual Reports*.

Only 10 of the 53 projects synthesized in Figure 2 were targeted research; that is, they were in response to SSHRC calls for research in specific areas. These targeted areas were: The Human Context of Science and Technology, Education and Work in a Changing Society, Women and Change, Women and Work, and Science Culture in Canada. Thus, the bulk of the research funded by SSHRC in mathematics education is independent research; that is, the specific areas of research are generated by the researcher rather than by the granting agency.

But there is other research in mathematics education being conducted in Canada than is made known by the SSHRC and FCAR *Annual Reports*. An obvious body of work is that found in masters theses and doctoral dissertations. Because of time limitations, it is not possible to give a complete, or even partial, overview of this research. However, we can point to a few examples that have been presented at the meetings of the Canadian Mathematics Education Study Group (CMESG). CMESG brings together on an annual basis Canadian mathematicians and mathematics educators, and provides a forum for researchers, old and new, to discuss their work. CMESG, which was set up in 1977, has as its "two main interests mathematics teacher education and mathematics education research, with subsidiary interests in the teaching of mathematics at the undergraduate level and in what might be called the psycho-philosophical facets of mathematics education (mathematization, imagery, the connection between mathematics and language, for instance)" (Wheeler, 1992, p. 5). A recent innovation in the annual program of CMESG meetings is the reporting of newly completed dissertation research.

Some of the current research that has been shared at CMESG annual meetings is the following (see the *Proceedings* of CMESG for the years 1993, 1994, 1995).

1993 Meeting:

O. Chapman, "Narrative inquiry in teacher research and teacher development."

R. Pallascio and R. Allaire, "Geometric spatial competencies among young Quebecers from the North (Inuit) and from the South."

E. Wood, "Connected knowledge in prospective secondary mathematics teachers."

1994 Meeting:

J. Barnes, "Enaction."

1995 Meeting:

G. Lynch, "The efficacy of an elementary mathematics methods course in changing preservice elementary teachers' mathematics anxiety."

And from the New Dissertations part of the 1995 CMESG annual program:

J. Barnes, "Enacting a chaos theory curriculum."

B. Davis, "Listening to reason: An inquiry in mathematics teaching--a report of my doctoral research."

G. Doctorow, "Writing to learn high school mathematics and conceptual growth."

Other conferences where Canadian mathematics educators disseminate their research work include PME, PME-NA, AERA, ICME, and so on; an examination of the titles of the work presented at the meetings of these groups yields further indication of the nature of the research areas being investigated by Canadians. But there is also some research being done that does not generally get reported at the already-mentioned conferences. This research is funded by school boards or ministries of education that do not provide for travel to meetings outside the geographic range of the funded project. For example, A. Taurisson, formerly of the Université du Québec à Montréal, was funded for several years by the Québec Ministry of Education in conjunction with a school board for research that involved the creation of software and the subsequent initiation of elementary school teachers into the use of computer technology in their teaching of mathematics. Reports of this work found their way into government publications and provincial teacher conferences. As an aside, the creation of software per se is not funded by SSHRC or FCAR unless it is embedded in a larger systematic inquiry.

At first glance, an examination of the various themes of the mathematics education research conducted by Canadians shows both theoretical and practical orientations. This is supported to a certain extent by the results of a survey carried out by Mura (1994). From

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the responses to a questionnaire sent out to mathematics educators in Canadian universities in the early part of 1993, she found that:

> As for the goal of mathematics education, some respondents were oriented more towards theory and others more towards action. Thus, at one end of the spectrum, there are mathematics educators whose main interest is, say, a theoretical understanding of how people learn mathematics and who are operating more or less within the territory of psychology or cognitive science, and, at the other end, there are those whose overwhelming concern is classroom practice and for whom mathematics education is almost identical with mathematics teaching. (p. 9)

Of Mura's 63 respondents who acknowledged mathematics education as their primary field and whose responses were retained for her analysis, 47 (75%) worked in education departments, 13 (21%) in mathematics departments, and 3 had joint appointments; 11 of the 13 from mathematics departments were from two Québec universities (Concordia and Université du Québec à Montréal) where mathematics education is housed in Departments of Mathematics. This latter fact translates into research questions in which the mathematical content is non-negligible. In fact, according to Figure 2, the areas of interest of a majority of researchers are more oriented toward the subject matter of mathematics and its learning than the mathematics teacher; the training of the mathematics teacher is a research preoccupation of less than 10% of SSHRC-funded mathematics education researchers.

But there is much more work in teacher development occurring than is reflected in the number of research grants being awarded in that area; however, it is not being pulled together into research proposals for funding consideration. Figure 1 indicated that certain provinces are rather under-represented when it comes to federal funding of mathematics education research. Mathematics educators in those provinces tend to be more involved with teacher education and with presenting their views at teachers' meetings. In fact, in certain provinces there has been a real tension between teacher training (professional development) and research, that is, between an emphasis on Master Teachers (who may not have more than a bachelor's degree) and those with research credentials. In these provinces, those who have been in a position to decide have put the emphasis squarely on teacher development.

Some Theoretical Orientations

Figure 2 provided a general indication of the mathematics education research areas of Canadians--learning, problem solving, use of technology, development of learning/understanding models, some teacher development, and so on. If one goes beyond the titles, the picture one obtains is much more varied. Unfortunately, it is not possible to do justice to the theoretical orientations of the 53 projects synthesized in Figure 2 in this brief panel paper. The most that can be done is to point to a few generic examples.

Because of the language of the majority of Quebecers, the French mathematics education literature has had a degree of influence. Thus, the *Recherches en didactique des mathématiques* (RDM) perspective can be found in certain Québec studies. According to Bartolini Bussi (1994) who has analyzed this perspective, "RDM aims at building a coherent theory of phenomena of mathematics teaching" (such as Brousseau's Theory of Didactic Situations) and "is oriented toward *knowledge of classroom processes*." (p. 123). In this tradition, the researcher acts as a *detached observer* of the situation. Some of our researchers have attempted more or less to follow this approach. Others have gone in a different direction with their studies of individual or small-group learning, often outside the classroom--showing the impact of Piagetian theory and methodology.

The U.S. research literature is also reflected in some Canadian studies. For example, the work of Cobb, Wood, and Yackel, in which the researcher is more of a *participant observer* and the teacher is a key decision-maker in the research process, has served as a model. Here the aim is often the production of learning tools and the subsequent analysis of their use and usefulness. Thus, in comparison with RDM-inspired work, not only does this approach address different problems and answer different questions, its method of inquiry is different.

Much Canadian research has tended by and large to be Piagetian in spirit. Constructivism, cognitive conflict, epistemological obstacles, the relative neglect of the role of the cultural tradition and social factors in the learning of mathematics--these have all characterized a great deal of our work. But the recent influence of Vygotskian theory, in which the process of learning is not separated from the process of teaching and is based on social relations, can now be seen in a few studies. In fact, some current programs attempt to interweave both theories. Such is the complex reality of research--to be Vygotskian during one phase of our research and Piagetian during another.

Other new work, which is also Piagetian in its roots, espouses *enaction*, a theoretical non-representationist approach developed by Varela, Thompson, and Rosch (1991), which Barnes (1994) has characterized in this way: "The world is not completely independent nor is it dependent. It is not purely the construct of our own thoughts or perception but is enacted by our structure. Thus it is not an independent entity 'out there' nor is it the internal construction of individuals. It is the interplay between the internal constructions and the structure" (p. 149). Still others are drawing from the hermeneutic tradition, a field of inquiry concerned with interpretation, which Davis (1995) has described as follows: "The hermeneut cannot maintain the detached objectivity of the scientist, for we are thoroughly implicated in the social phenomena we study. The essential point here is that in the simple act of investigating mathematics teaching, I and my collaborators in research are helping to shape the phenomenon. As our ideas change, our actions change. What mathematics teaching *was* when we began our study is not the same as what it *is* now" (p. 159).

Closing Remarks

There is no single "world view" among Canadian mathematics education researchers. We are basically quite eclectic, borrowing from here and there, according to our individual tastes. And there is no indication that this variety of perspectives will decrease; in fact, the recent doctoral dissertation research that has been presented at CMESG suggests that the variety of perspectives is increasing. I believe that, in the future, we will also be seeing more research reflective of the social and cultural emphases of Vygotsky, but because so much of our research is mathematically driven (see Bishop, 1992), it is quite unlikely that we will lose the mathematics in the shift. Nevertheless, a good reminder to us as we consider taking on more socio-culturally oriented work is that provided by Bartolini Bussi (1995) in her panel contribution at the PME conference in Recife, Brazil. She was quite critical of the non-interventionist nature and subsequent socio-cultural analysis of the protocol excerpt under examination by the panel in that it spoke much more to social issues than mathematical ones, and concluded by asking: "But what mathematics did the children actually learn?"

It is not just in their theoretical frameworks and methodologies that one Canadian mathematics education researcher can be quite different from another. As I mentioned earlier, there is no national curriculum. Since both the mathematics curriculum and educational approaches for dealing with that curriculum vary from province to province, there is no coming-together of mathematics education researchers to solve national problems. The non-centralized nature of the Canadian scene thus produces less of a common focus and, when compared with a country having a national approach to the mathematical education of her students, several differences in the ways that research is mobilized and oriented. All Canadian mathematics education researchers would likely agree that their goal in doing research is to better understand the teaching and learning of mathematics, and thereby possibly help to improve it; but after having said that, there is no singular motivation for doing their research. Each researcher or team of researchers decides on the problem to be investigated; and if the granting agency with the help of its pool of both national and international reviewers finds that this problem is not worth investigating, it won't be funded.

In a sense, it is precisely because of the lack of a clear Canadian identity and the built-in decentralization of our federal structure that we tend to turn outward rather than inward--at least until very recently. Putting aside for the moment the current political mess in which the country finds itself, the formation of PME in the late 1970s fit very nicely with the felt need of many Canadian mathematics education researchers both to know more about what was going on internationally and to attempt to broaden their own perspectives. We are, in fact, a quite loosely knit group of individuals who respect and have great fondness for each other, but who may have as much, or more, in common with international research colleagues than with our national ones. It is really only the Canadian Mathematics Education Study Group (CMESG) that provides a national vehicle for bringing together Canadian mathematics educators; but this association is not for researchers only. As mentioned earlier, its annual meetings of about 50-60 persons include some mathematicians, some mathematics educators who do research, and some who don't. There are some mathematics education researchers and many mathematicians who do not attend CMESG meetings at all. Thus, the contacts among mathematics education researchers and, even more so, between mathematics education researchers and mathematicians in Canada can be quite tenuous indeed.

Just as it is very difficult, if not impossible, to define Canadian culture, it is equally challenging to do so for the Canadian mathematics education research culture. The term "culture" suggests a sharing of something. Despite our diversity, we do share a love of mathematics, an intense need to know more about how mathematics is learned and taught, and careers that are focused on the conducting of research that will enlighten us in some aspect of this domain. However, we do not share a common orientation as to the theoretical framework that shapes the research, the questions to be posed, nor the methodology to be used; hence the eclecticism of our activity.

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Appendix 1

Mathematics education research projects funded by SSHRC from 1985 to 1995 (titles in French have been translated to English; co-investigators are named in parentheses; [T] indicates targeted research)--Source: SSHRC *Annual Reports*.

1994-1995:

D. Biron, "Study of external representation activity in arithmetic and geometric problem solving."

R. Pallascio [T], "Geometric spatial competence and Inuit mathematical acculturation" (R. Allaire, L. Lafortune, P. Mongeau)

L. Poirier, "Implicit models used by children of different school levels in solving complex additive problems."

S. René de Cotret, "Validation processes in algebraic problem solving at the beginning of the secondary level."

W. Rogers, "Improving assessment in junior high school mathematics" (T. Maguire, R. Mulcahy, S. Sigurdson, S. Norris).

V. Zack, "How is mathematical meaning shared? An investigation of explanations by grade five children during small-group problem-solving discussions."

1993-1994:

A. Geddis, "Teaching and learning about teaching" (B. Onslow).

J. Giroux, "Construction of knowledge on numeration and operations in kindergarten and first grade pupils."

C. Kieran, "Learning and teaching secondary school algebra: Comparing process-oriented and object-oriented approaches" (A. Sfard).

T. Kieren, "Portraits of mathematical understanding: Extended investigations into a theory for the growth of mathematical understanding."

N. Nantais, "Study of the notion of division in the construction of multiplicative structures in elementary school children."

A. Sierpinska, "Undergraduate Linear Algebra: Developing algebraic sense in students" (J. Hillel).

R. Zazkis, "Learning mathematics by preservice elementary school teachers."

1992-1993:

N. Bednarz, "The emergence and development of algebraic reasoning and representations" (B. Janvier).

S. Lajoie, "A computer-based learning environment to promote scientific reasoning."

D. Maclean, "Encouraging young women's participation in mathematics and science: A developmental perspective" (V. Shuart, D. Keating).

1991-1992:

J. Brine [T], "Adjusting to the changing context for computers in education: New perspectives on technology in schools" (M. Johnson, D. Robitaille).

N. Herscovics, "Development and experimentation of constructivist models to describe the understanding of transcendental functions in high school mathematics" (J. Bergeron).

D. Sawada, "The contribution of a literary approach to the study of mathematics in education."

1990-1991:

A. Boileau, "The use of computer environments in the learning of algebra" (M. Garançon, C. Kieran).

C. Gaulin, "Levels of understanding of the processes of simulating random phenomena."

J. Hanrahan, "An analysis of the strategies used by intellectually handicapped children when learning to add small numbers" (E. Lusthaus, S. Rapagna).

C. Janvier [T], "Contextualized reasoning and professional training" (M. Baril).

T. Kieren, "Investigations into a recursive theory of mathematical understanding."

D. Owens, "Understanding children's development of rational number concepts."

1989-1990:

N. Bednarz, "A didactical intervention: A socio-cinematic approach to the problem of representing dynamism in mathematics" (C. Garnier).

J. Bergeron, "Integration of the notions of cardinality and ordinality in the construction of number in 5- to 7-year-old children" (N. Herscovics).

M. Coron, "Spatial representations of elementary school children."

C. Dassa, "Elaboration of a model integrating pedagogical diagnosis of school learning and its application to the first cycle of high school mathematics" (D. Ajar, C. Parent, S. Seguin).

J. Hillel, "Using computer algebra systems with adult learners."

J. Lajoie [T], Computer-training of visual-spatial abilities and problem solving."

B. Randhawa, "Mathematics competence and personal, social, and cognitive constructs" (J. Beamer).

1988-1989:

A. Boileau, "The use of computer models in the learning of algebra" (M. Garançon, C. Kieran).

C. Gaulin, "Understanding the simulation of random phenomena."

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I. Kleiner, "The history of mathematics and its relevance for teaching."
P. Rogers [T], "Increasing participation by women in mathematics and science."
S. Sigurdson, "Meaning in mathematics teaching" (A. Olson).
D. Owens, "Thinking on rational number concepts: A teaching experiment" (S. Vance).

1987-1988:
M. Bélanger "A study of children's and adolescents' use of mathematical concepts and representations in solving three-dimensional space problems posed in a computer directed micro-world" (J.-B. Lapalme).
J. Hillel, "A Logo environment to enhance an analytical solution schema by 12-year-olds" (C. Kieran).
E. Kuendiger, "Pre-service teachers' beliefs about achievement in mathematics and about effective teaching of mathematics" (L. Pereira-Mendoza, E. Williams).
R. Mura [T], "The expression of subjectivity in scientific texts" (M. Duncan).
R. Pallascio, "The effect of environment in the organized development of perception of geometric objects" (R. Allaire).
P. Rogers [T], "A study of a successful model for recruiting and training female mathematics students at the post-secondary level" (J. Poland).
D. Wheeler, "Algebraic thinking in high school students--Part II: Some effects of instruction and the role of language."

1986-1987:
R. Allaire, "Development of perception of geometric objects in a computer environment."
N. Bednarz, "Study of representations developed by children in a mathematical problem-solving context involving dynamic concepts" (B. Janvier, J. Wallot).
D. Owens, "Decimal concepts and operations: A teaching experiment."
D. Wheeler, "Algebraic thinking in high school students: Their conceptions of generalisation and justification."

1985-1986:
T. Kieren [T], "Micro-electronic technology and Canadian youth: A study of access, values, and social issues" (W. Egnatoff, W. Higginson, D. Thérien, G. Jeffrey, W. Spain).
I. Kleiner, "The history of mathematics and its relevance for teaching."
R. Mura [T], "Mathematics, women, and culture."
D. Wheeler, "Algebraic thinking in high school students: Their conceptions of generalisation and justification."

Appendix 2

Mathematics education research projects in Québec funded by FCAC in 1978-1979 (French titles have been translated to English)--Source: FCAC report: *Les subventions de formation de chercheurs et d'action concertée. Crédits alloués: "Équipes et séminaires," 1978-1979.*

N. Bednarz, "Elaboration and evaluation of learning situations in elementary school mathematics."
M. Bélanger, "Didactical problems in the learning of mathematics at the elementary school level in underprivileged milieus."
V. Byers, "Validation and application of a model of mathematical understanding."
C. DeFlandre, "A multi-disciplinary perspective on the analysis of the learning of mathematics by means of problem solving among children from 5 to 14 years of age in underprivileged and regular milieus."
J. Hillel, "Aspects of problem solving."
D. Lunkenbein, "Jean Piaget's concept of *groupements* as a rationalization tool in didactical interventions."
E. Quintin, "Elaboration of a 'readiness for the learning of arithmetic' plan: Its effects on a 'normal' and 'learning handicapped' population."

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THE RESEARCH CULTURE OF U.S. MATHEMATICS EDUCATION[1]

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Abstract: An attempt to locate and characterize a culture of research in U.S. mathematics education begins with a description of the community sharing that culture. Trends in U.S. research are discussed, but seem not to set it apart from that in other countries. Instead, the U.S. research culture is seen as stemming from character traits often used to describe the United States more generally. Although criteria of quality remain uncertain, much U.S. research meets the definition of *culture* as excellent work.

The original title of this paper had *cultures* in the plural form. My intention was to describe several cultures of research practice within U.S. mathematics education. As I thought about the topic, however, I realized that the similarities of practice among members of what might be called the research community in mathematics education in the United State are much stronger than the differences. I decided to use the singular term. Then I became concerned because I saw that if one could put a fence around the United States, it would be difficult to argue that any "research culture of mathematics education" within that fence was distinctively different from that of people outside. National borders, especially today, are seldom indicative of cultural boundaries. Nonetheless, I attempt in what follows to consider some features of the U.S. research community in mathematics education and how they might, in some sense, be seen as yielding a common culture.

An examination of a research culture needs to begin with a consideration of what *culture* might mean in this context. One view of culture (Geertz, 1973, p. 5) refers to the "webs of significance" we have spun for ourselves, the meanings we have created for what we do. We share a culture as researchers in mathematics education to the extent that we see ourselves involved in an enterprise that we interpret in the same fashion. Researchers in mathematics education in the United States are engaged in a variety of activities that they term *research*. Although a critic might claim that some of those activities do not deserve the label, I think it is fair to say that there is today a broad consensus across the community on what counts as research (even though there might simultaneously be considerable uncertainty and disagreement as to how its quality is to be judged).

Another view of culture refers to a set of people with common beliefs, common social forms, and a common way of life. Under that definition, the United States has a research culture to the extent that a group of researchers can be identified who share certain commonalities. To take a highly inclusive view, one might say that the culture of research in mathematics education comprises everyone who has, at one point or another, engaged in systematic inquiry into a topic potentially related to the teaching or learning of mathematics. That group would be quite large. It would range from the master's degree student in a department of curriculum and instruction who chooses elementary arithmetic as a research focus, to the educational psychologist studying spatial reasoning, to the senior scholar in mathematics education conducting ethnographic studies of mathematics classrooms, to the sociologist examining the uses of mathematics in daily life, to the government bureaucrat or educational historian analyzing course-enrollment data. The difficulty with this too-broad interpretation is that it does not describe a community of common interest and identity.

An alternative interpretation is to look at how people identify themselves. Do they see themselves as mathematics educators engaged in research, or do they separate themselves from that community in some fashion? For example, many U.S. psychologists use mathematics as a vehicle for their research into learning and instruction. Are they therefore a part of the research culture in U.S. mathematics education? My answer is no. Valuable though their work might be to teachers, policy makers, and other scholars, they are not members of the mathematics education community in the United States and therefore not participants in its research culture.

Others have remarked on the schism between experimental psychologists working on children's mathematical development and researchers who might be identified as scholars in mathematics education (Davis, 1996; Geary, 1994). The psychologists tend to take the school mathematics curriculum as unproblematic, unlike the majority of mathematics educators engaged in research. Many educational psychologists conducting research on the learning of mathematics are concerned largely, if not exclusively, with computational arithmetic, algebraic manipulation, or the solution of standard word problems. They emphasize the development of skilled performance with symbols and often have a rather different orientation than most mathematics educators with regard to issues of reform in curriculum and instruction (Penner, Batsche, Knoff, & Nelson, 1994).

The community of researchers in mathematics education in the United States possesses a common culture with regard to research because its members have been socialized into the U.S. educational system and operate within and across similar institutional structures. They identify themselves, first and foremost, as mathematics educators and regard themselves as engaged in mutually supportive, collective action (as indicated by the numerous attempts over the past three decades to lay out research agendas). The community has a shared vision of the problematic nature of mathematics teaching and learning, as well as of the power of disciplined inquiry to improve the educational process. The special qualities of this research culture can be seen as stemming from what might be called the U.S. character.

The U.S. Research Community

The community of researchers in mathematics education in the United States is a key part of a community of mathematics educators, who in turn are part of the U.S. mathematics community. For over a century, there has been in the United States a variety of links between people concerned with the creation and uses of mathematics and those concerned with its teaching and learning. They have belonged to sister organizations, attended joint meetings, and read a common set of journals and other publications. Relations have at times been strained, and there have been periodic misunderstandings and differences of opinion. But the U.S. mathematics community has been remarkably unified when compared with similar communities in other countries or in other academic disciplines. Organizations such as the Conference Board of the Mathematical Sciences and, more recently, the Mathematical Sciences Education Board have enabled mathematicians and mathematics educators to engage in a mutual dialogue and to speak with a common voice on matters of public policy. Especially important to the welfare of the mathematics educators has been a virtually continual stream of support from prominent research mathematicians who not only maintained an interest in educational matters but were willing to immerse themselves from time to time in the politics of education.

Consequently, as the research community developed in mathematics education over this century, it developed under the aegis, if not always the understanding or even support, of a larger community. The research community has had to fight various battles to become accepted, but once accepted, it has benefited from the strengths of the larger group.

The heart of this community might be said to consist of those U.S. researchers who routinely report their work at meetings of PME, PME-NA, or the Special Interest Group for Research in Mathematics Education. Many of them hold offices in these groups. They appear in sessions devoted to mathematics education research at annual meetings of the American Educational Research Association and of the National Council of Teachers of Mathematics. They publish in the pages of the *Journal for Research in Mathematics Education*, the *Journal of Mathematical Behavior*, and, increasingly, *Educational Studies in Mathematics*. They serve on committees such as the Research Advisory Committee of the NCTM.

Much of the research done by members of this group is conducted as part of projects supported by government or private foundations. These projects allow researchers from different institutions to engage in collaborative work and have, over the years, made the community more cohesive. The projects are ordinarily aimed at the development of curriculum materials or at teacher education. Consequently, the research activities are subordinate and often tangential to the main enterprise. The influence of the funding agencies on the nature and direction of the research is a subtle one that has yet to receive serious study. There is no doubt, however, that many U.S. researchers in mathematics education have followed the winds of political fashion.

The great majority of mathematics educators in the United States who have conducted research beyond that needed for a doctorate are engaged in teacher education in institutions of higher education. Some are members of departments of mathematics in colleges of arts and sciences; many more are members of schools or colleges of education, serving in departments of curriculum and instruction, of teacher education, or of science and mathematics education. Only a handful work within centers or institutes where the majority of their time can be given to research. The rest turn to project funding to buy time away from teacher education activities so that they can do the research needed to achieve tenure and promotion. In recent years, some university-level researchers in mathematics education have enlisted practicing teachers in their research projects, responding to the argument that teachers are part not only of the community of mathematics educators but also of the community of researchers.

At many U.S. institutions of teacher education, there may be only one or two mathematics educators with a serious interest in research. Despite the relative openness of the research community to new ideas and new members, it is ordinarily quite difficult for these relatively isolated researchers to mount projects by themselves or join in the work of others. When the Georgia Center for the Study of the Teaching and Learning of Mathematics was established in the 1970s with (modest) support from the National Science Foundation, it managed to tap into a deep vein of talented people working more or less alone at colleges across the country who needed only some contact with colleagues to get them started on research. The National Center for Research in Mathematical Sciences Education at Wisconsin subsequently used a similar model to bring together so-called invisible colleges of researchers. The NCRMSE, however, concentrated less on isolates and more on a broad spectrum of researchers, including many from outside the national community of mathematics educators.

The U.S. research community in mathematics education can be seen, then, as comprising a nucleus of active, committed scholars engaged in mutual dialogue and many collaborative activities plus a large body of relatively isolated scholars who maintain contact with the rest by attending meetings, reading research reports, and occasionally joining in the dialogue. Outside the community are many other researchers whose work may contribute to the dialogue but who do not take U.S. mathematics education as their primary professional interest.

Trends in U.S. Research

Much of the current culture of U.S. research in mathematics education is reflected in trends in how research is conducted and what questions are being investigated. Most of these trends are international in character. They are often more pronounced, however, within the United States, partly because the huge volume of work that is done there allows for greater variation and therefore more extreme cases.

Educational research in general within the United States has moved dramatically over the past two decades from a preoccupation with quantitative methods to a pervasive use of qualitative methods. The controversies this change has engendered can be found in the pages of the *Educational Researcher* and elsewhere. Courses in qualitative research methods have proliferated within colleges of education, and enrollments have fallen in courses dealing with the statistical analysis of data. The change arose from a rejection of experimentation in the physical sciences as the ideal for educational research, and was accompanied by the abandonment of a heavy reliance on techniques borrowed from experimental psychology and psychometrics.

The change can be said to have began at the same time and perhaps even earlier in the U.S. mathematics education research community, which was never as fascinated with experimental studies as were other educational researchers. Perhaps because of their awareness of the assumptions needed to fit statistical models to real data, most U.S. researchers in mathematics education tended to be skeptical of statistics even when they were using them. In a recent paper, Nerida Ellerton and Ken Clements (1996) have claimed that two highly influential books characterized as promoting a "scientific" approach to research appeared in the United States at the end of the 1970s, the books by Begle (1979) and by Shumway (1980), and managed to "straitjacket" thinking about research within the international community of mathematics educators for years. Whatever the view from outside, the prevailing view within the U.S. community of mathematics education researchers at the time was that both books reflected a too-narrow vision of research and were in some sense outdated before they were published.

Since then, U.S. researchers in mathematics education have embraced qualitative research methods wholeheartedly and often rather uncritically. They have dropped not only inferential statistics but also descriptive statistics from most of their research reports. Largely abandoning attempts to compare different methods of teaching the same content, they have turned to case studies of students and teachers and ethnographic studies of classrooms.

Accompanying and encouraging the shift in methodology has been a rise of interest in teaching practice and classroom discourse. The principal focus of research has moved away from the individual learner and toward the teacher engaged with a classroom full of learners. Teachers' views of teaching, of learning, and of mathematics have received greater attention as researchers have attempted to see these views reflected in the teachers' practice. Teaching, learning, and mathematics itself are seen as situated in social contexts that determine what is both taught and learned. The emphasis has been on interpreting the meanings that teaching and learning mathematics have for the participants in the process.

In contrast to developments in some other English-speaking countries, there has been no great surge in research based on critical theory within the U.S. mathematics education community, nor has there been much of what has been termed "action research."

In a recent effort to look into the future of research in mathematics education (Silver & Kilpatrick, 1994), researchers in several countries mentioned the learning of collegiate mathematics, classroom microprocesses, schools as settings for learning, changes in teacher professionalism, the diversity of student populations in schools, the use of technology in mathematics instruction, and the assessment of mathematics learning as topics likely to receive increasing attention from researchers. All of these are on the agenda of U.S. researchers in mathematics education. U.S. research trends both lead and follow what is happening elsewhere.

The U.S. Character

What, then, makes U.S. research in mathematics education distinctive? Where is the U.S. research culture? My answer is that it can be seen in aspects of the U.S. character that are manifested in how we do, think about, and talk about our research. These character traits may be stereotypic and therefore to some extent unfair, but collectively they may reveal how our research can be set apart from that done elsewhere.

The United States is a big, wasteful, materialistic, acquisitive, competitive nation. No other country has asserted publicly its intention to be "Number 1" in the world in science and mathematics. Nowhere else have politicians heaped such scorn upon their public schools while simultaneously linking their country's standing in the world to their children's knowledge of mathematics (and science). In the country's political discourse, the aims of education are to yield personal and corporate economic advantage. Little or no mention is made of mathematics education as having social, intellectual, aesthetic, or political aims.

The U.S. mathematics education community has managed to turn this situation to their profit. By promoting higher "standards" for mathematics education, they have allowed politicians to buy into education reform at essentially no cost, creating some close alliances between policy makers and the mathematics education community. Under this umbrella of political support during a time when government funding is increasingly difficult to secure, researchers in mathematics education have been able to keep some of their projects alive and even begin new ones, particularly when they can invoke the promise of technology. What happens when reform fails to materialize and the U.S. fails to be Number 1 in mathematics education has yet to be reckoned with. Fortunately, perhaps, the national attention span is rather short.

The U.S. temperament is pragmatic. It is oriented toward action and not theory. From research in education, it wants to know "what works." Always anti-intellectual, it has in recent years become skeptical of science even as it profits from scientific advances and encourages its children to learn more science. U.S. research in mathematics education has never been strong on theory use or theory construction, but lately it has moved away from any pretense of being "scientific." Its current preoccupation is with understanding specific phenomena, largely through a personal account, and not with arriving at explanations or generalizations.

Always present-oriented, the United States has little interest in what happened yesterday. It has tended to romanticize its past and to see the present as somehow the end to which all history has been aimed. Moreover, it has periodically been swept by waves of religious reawakening, when lost souls are told that the millenium is at hand. The current "constructivist" movement in U.S. mathematics education fits that pattern.

In the popular mythology of constructivism, U.S. mathematics teachers, until well beyond the middle of the 20th century, thought of children's minds as blank slates to be written on or as empty vessels to be filled. They did not realize that children entered their classroom with ideas about what they would learn. The demon responsible for this epistemology was E. L. Thorndike, who through his behaviorist philosophy was responsible for mathematics textbooks filled with repetitive exercises and classrooms echoing with mindless drill. Never mind that Thorndike wrote with some sensitivity about meaning, understanding, and relational thinking in mathematics, or that John Dewey had early in the century brought notions of reflective thinking about the solution of practical problems into many U.S. mathematics classrooms, or even that Plato had something to say long ago about how mathematics might emerge out of dialogue. History began when Jean Piaget showed us child's minds at work (more precisely, it began in the 1960s when Piaget had been translated into English and began to be discussed in U.S. colleges of education), and it reached its end for U.S. mathematics educators when Ernst von Glasersfeld (1991, 1995) proposed a "radical" version of constructivism. Today, any U.S. mathematics teacher who is using concrete materials in instruction or whose students are working in groups is likely to think of herself as engaged in "constructivist teaching" and U.S. researchers apply the constructivist label to almost everything they do.

Fifty years from now, historians will no doubt wonder why and how the U.S. community of researchers in mathematics education got so caught up in radical constructivism in the late 1980s and early 1990s. What von Glasersfeld has termed "trivial constructivism," in which one accepts little more than that students actively construct their knowledge, is widely accepted by educators today throughout the world, often under a different label than "constructivism," but U.S. mathematics educators seem to have had a particular affinity for the radical version in which the known world is seen as a mental construction having no objective reality. That version has not received much serious criticism from within the U.S. community of mathematics educators, despite the increasing number of critiques from outside (e.g., Lerman, 1996; Marton & Neuman, 1989;

Matthews, 1994; Osborne, 1996; Phillips, 1995; Zevenbergen, 1996; for "insider" critiques, see Kelly 1995; Orton, 1995). Its internal contradictions—principally its professing of agnosticism regarding ontology while adopting a nonrealist ontology at the same time—have largely been ignored by U.S. researchers, as have its claims from cybernetics that learners are informationally closed systems.

Instead, a spirit of triumphalism has prevailed. Paul Ernest (1995) claims that radical constructivism was launched at the PME meeting in Montreal in 1987 [he says 1983] "to widespread international acceptance and approbation" (p. xi). We are told (Steefe & Kieren, 1994) that there has been a constructivist "revolution" in mathematics education "of a magnitude no less than the modern mathematics movement of the 1960s" (p. 720). Constructivism, presumably of the radical sort, is seen as having influenced the reform movement in U.S. mathematics instruction even though few of the people behind those reform efforts would claim to be radical constructivists, and the constructivist language used in reform documents falls into the category of what is now more charitably termed "realist constructivism" (Cobb, 1994) rather than "trivial." Radical constructivists dismiss critics as "superficial or emotionally distracted" (von Glasersfeld, 1991, p. xv) or as victims of "Cartesian anxiety" (Steffe & Kieren, 1994, p. 723).

What is curious, I repeat, is how U.S. scholars in mathematics education have been caught up in this movement. (Of course, others outside the U.S. are caught up too, primarily in Canada, the U.K., and Australasia, but the movement seems more firmly rooted in the U.S.). Outside the mathematics education community, even within the United States, constructivism is seen as just one of a number of perspectives to be considered when dealing with educational issues. For example, Jerome Bruner (1996) in advancing nine "tenets" that ought to guide what he terms "a psycho-cultural approach to education," devotes only 11 lines in 28 pages of discussion (pp. 13-42) to constructivism. In her review of recent approaches to knowledge and their influence on educational research, Maxine Greene (1994) makes no reference whatsoever to radical constructivism. Contrast that with the volume of constructivist discourse flooding the research literature and research meetings of U.S. mathematics education.

Constructivism seems to be, for most U.S. mathematics educators engaged in research, not a theoretical position to be elaborated, tested, or critiqued but instead a bandwagon to be followed. The so-called back-to-basics movement (not really a movement at all but itself little more than a slogan) that followed the new math era in the United States scared many mathematics educators into thinking that the days of Thorndike and of drill and practice were back. They responded by attempting to redefine "problem solving" as a basic skill. Once problem solving had come and gone as a slogan, constructivism was there to fill the void. U.S. mathematics educators need some means to justify their continued belief that teachers should not be at the front of the classroom lecturing to students about mathematics.

As a required school subject that many students find difficult and even repellent, mathematics poses special problems for educators and seems to call for special solutions. U.S. mathematics educators are beset by claims that students elsewhere are learning more and better mathematics. Constructivism has provided them with a refuge.

Paul Cobb (1995) argues that the old "conceptualist" version of constructivism is gone and that constructivism is "an evolving paradigm." He does not seem to be referring to van Glasersfeld's version of radical constructivism, which has been quite consistent over the years and is still with us. Efforts to explain how, within an individualistic radical constructivism, we can have intersubjectivity and come to know other minds, however, have not been widely accepted. Increasing numbers of U.S. researchers in mathematics education seem to be abandoning the label of "radical" and are now claiming to be "social constructivists," which may be what Cobb means by evolution. The challenge to radical constructivism, as Cobb (1994) has pointed out, is to show how cognition is socially and culturally situated. Observers such as Robyn Zevenbergen (1996) see the social implications of the construction of meaning as a problem for constructivism as a whole. Perhaps the U.S. research culture in mathematics education is at last beginning to awaken from its constructivist dream.

The United States was built on the spirit of "rugged individualism," but there has always been a complementary commitment to community building. The current turn toward the classroom as a social system and to mathematics as culturally determined and socially situated may mean that U.S. mathematics educators are ready to address more seriously than they have in the past issues of language and cultural difference. Of course, that will take place in the U.S. context, which tends to be highly legalistic about such matters. All of the Americas are multi-ethnic and multi-cultural, but only in the United States are people so prone to use lawsuits to accomplish social ends. An unstudied problem in U.S. mathematics education is the extent to which legal barriers denying access to children and classrooms may increasingly be driving researchers to look to collegiate mathematics and teacher education as sites for their work. In may also be the case that some researchers are enlisting practicing teachers as colleagues as a way of gaining such access.

Foreign visitors to the United States often remark on how insular it is, how uninterested people are in what is happening elsewhere in the world, unless it directly impinges on the U.S. "national interest." U.S. research in mathematics education also shares that character (Bauersfeld, 1992). U.S. scholars are relatively unfamiliar with work in mathematics education published outside the United States unless it is published in English and often not even then. Mathematics educators in other countries look to the United States for ideas on research, but U.S. mathematics educators do not reciprocate. They see themselves as having no need to learn a foreign language, and when they travel abroad they are often more interested in telling than learning. That attitude may be changing. Even as U.S. politicians move to make English the country's official language, U.S. researchers in mathematics education seem increasingly aware that good research is being reported in languages other than English.

One can reasonably hope U.S. mathematics education will soon achieve a more balanced perspective on research and the variety of ways it can be conducted. Difficulties of judging the quality of our research are apparently becoming more pronounced (as indications, see Silver & Kilpatrick, 1994; Lester, 1996), and the community may be able to use the current uncertainty about standards for research as an occasion to achieve a better understanding of research itself. Mathematics educators can certainly benefit from research that provides interpretations of teaching and learning, but they can also benefit from research that provides explanations. There is still a place in our work for theory and for evidence to support it.

Culture as Excellence

One common meaning of *culture* refers to what is considered excellent in matters such as the arts, literature, manners, or scholarly pursuits. Under that interpretation, some have suggested that the phrase "U.S. culture" is an oxymoron. When one looks at how the United States is defined by its junk food, movies, and music videos around the world, some skepticism may be justified. But in that same sense, the U.S. does have a research culture in mathematics education because so much high quality work has been done here over the years, as the recent *Handbook of Research on Mathematics Teaching and Learning* (Grouws, 1992) demonstrates.

Much of the best in U.S. culture has come from the arrival on our shores of talented people from other lands. We are indeed a nation of immigrants, and those immigrants have always played key roles in elevating our culture. They have supplied much of the energy and optimism that help to define us. An example of the best in the U.S. research culture in mathematics education is provided by Alba Gonzalez Thompson, who died suddenly and unexpectedly in August. Alba's family came to the United States to find freedom, and she took the opportunities she found here to forge an exemplary career as a researcher. Her thoughtful work on teachers' conceptions and beliefs about mathematics is notable both for its scholarly depth and its strong connections to practice. It was marked by intellect, elegance, and originality. Though cut short by her death, it will always stand as part of our research culture.

The United States is a diverse, changeable, contradictory place that observers have found hard to describe. Anything that might be called its *culture* is likely to look different from another vantage point. I have tried to sketch what the research culture in U.S. mathematics education looks like from one perspective. I am aware that I have overlooked much that is important and that I have not been able to provide much detail to support my claims. I hope, however, that I have stimulated some thought on what makes our culture so distinctive.

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ADVANCED MATHEMATICAL THINKING

FIRST-YEAR CALCULUS STUDENTS' PROCEDURAL AND CONCEPTUAL UNDERSTANDINGS OF GEOMETRIC RELATED RATE PROBLEMS

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In this study, subjects' abilities to solve geometric related rate problems were investigated, the procedural steps which gave subjects the most difficulty were identified, and subjects' thinking about changes in geometrically related variables was documented. Two written tests and an interview were administered to college students enrolled in first-year calculus. Of the 154 subjects in the sample, six participated in think-aloud interviews, in which they determined how the growth of linear dimensions affects volume, area, and perimeter. Strategies used in the interviews were identified as process-oriented or object-oriented and were used to construct a developmental scale of understanding of the concept of changes in related dimensions. Robust solvers of geometric related rate problems advanced to object-oriented strategies, while weak solvers of geometric related rate problems did not.

Statement of the Problem

The purpose of the study was to assess first year calculus students' abilities to solve geometric related rate problems, determine their facility with the underlying procedural skills, and characterize the conceptual understandings of weak and robust solvers. *Geometric related rate problems* are those in which the instantaneous rate of change of one variable, representing one measurement of a geometric figure, is given and the instantaneous rate of change of a related variable, representing another measurement of the same geometric figure, is requested. The context of geometric related rate problems was chosen because students' difficulty with the application has been documented (White and Mitchelmore, 1996. Balomenos, Ferrini-Mundy, and Dick, 1987) and the context provided a vehicle for exploring subjects' understandings of a number of higher level topics: representation, dimension, and the functional relationship between changes in variables. Tests and in-depth interviews were used to gather evidence for subjects' robust or weak understandings.

Sample and Methods

Subjects were drawn from two Calculus I classes taught by different instructors during the fall semester of 1993 at a large urban university in the northeast United States. In Class A, students were primarily liberal arts or engineering majors. Class B was a special section of Calculus I for engineering majors who had performed poorly on a mathematics placement test administered by the university prior to matriculation.

Tests 1 and 2

The goal of Test 1 was to classify subjects as robust or weak solvers of geometric related rate problems. Test 1 consisted of three geometric related rate problems that varied by the dimension of the space measured by the related variables, the number of steps required to solve the problem, and the geometric shape that was expanding or contracting. The three questions on Test 1 were scored separately. Scores included partial credit for recording the relevant geometric formula, implicitly differentiating the geometric formula with respect to time, substituting specific values of the variables into the related rate equation, solving for the desired rate, interpreting the solution, and, where necessary, solving an auxiliary problem. Test 1 was taken by 115 students in Class A and 39 students in Class B.

The purpose of Test 2 was to determine the subjects' abilities to perform the procedural skills required to solve geometric related rate problems independently from the other skills. These skills were identified by the researcher and confirmed by experts. Test 2 contained items for each of the following skills: sketching and labeling a figure; translating facts in a problem statement to symbolic representations; recording geometric formulas; implicitly differentiating geometric and non-geometric formulas with respect to time; substituting values into a related rate equation and solving for a specified rate; interpreting computations that followed a problem statement; and solving simple geometry problems similar to the auxiliary problems in related rate questions (Faber, Freedman, and Kaplan, 1986). There was no partial credit; items were scored as correct or incorrect. Subjects received a subscore for each of the Test 2 skills. Each subscore was interpreted as the indicator of the subject's ability to perform that skill. Test 2 was taken by 34 students in Class A and 24 students in Class B.

Clinical Interview

The interview was conducted to obtain evidence of subjects' conceptual understandings of the relationships between changes in geometrically related variables. Interview subjects were selected from the 58 subjects who had taken both Tests 1 and 2. The Test 1 scores were distributed among three distinct groups: low, middle, and high. These groups were evident in a histogram of Test 1 scores, which appeared trimodal. As expected, the total scores on both Tests 1 and 2 were positively correlated. Six subjects were interviewed: two robust solvers, high scorers on both Tests 1 and 2 with Test 1 scores at or near the high mode; two weak solvers, low scorers on both Tests 1 and 2 with Test 1 scores at or near the low mode; and two outliers with great disparities between their Test 1 and Test 2 scores.

In the interviews, subjects were asked to think aloud while they performed three tasks (box, can, and ball), each comprised of three subtasks (sand, paper, and string). For the first task, subjects were asked to imagine a small box, filled with sand, wrapped with paper and tied with string, and to determine how many times as much sand, paper, and string would be required to fill, wrap, and tie a larger box whose dimensions were twice those of the small box (Banchoff, 1990). For the second and third tasks, subjects were asked to answer similar questions about a can and a ball. By examining an object between two static states, rather than in the dynamic state in which related rate problems occur, subjects' understandings of related changes in geometrically related variables was explored without the complication of the instantaneous change concept. During the interview, non-altering verbalizations were used to encourage subjects to continue thinking aloud, without influencing the solution. Altering verbalizations (prompts and probes) were used to assist subjects when they became stuck and to gather additional evidence of their conceptual understandings. Prompts helped guide the subject to the solution, while probes requested justification of steps and detailed explanations.

Results of Quantitative Analyses

Test 1 scores served as the indicator of subjects' abilities to solve geometric related rate problems. The mean for all subjects who took Test 1 (n=154) was 13.18 out of a possible 30 points (or about 44% correct), with a standard deviation of 6.62 points. A t-test determined that the means for the two classes were not significantly different at the 0.05 level (t=1.81, p=0.07). The mean scores, as percents, for all subjects on each of the three individual problems on Test 1 were 57%, 52%, and 28%, respectively.

Test 2 results documented subjects' abilities to perform each of the individual steps required to solve geometric related rate problems. The mean percent correct for all subjects on the items related to each step were: sketch and label a figure (17%); translate facts in problem statement to symbolic representations (27%); recall geometric formulas (78%); perform implicit differentiation (44%); substitute rates into equation and solve for the desired rate (41%); interpret results of solved equation (47%); and solve an auxiliary problem (53%).

Results of Qualitative Analyses

There were two main phases involved in the completion of each subtask in the interview. In the first phase, the subject reported the correct relationship for sand, paper or string. In the second phase, the subject was asked to justify that the Phase I solution was general or to generate and justify a general solution. Analyses of interview transcripts revealed that subjects used a variety of strategies in Phase I (Table 1) and a variety of justifications in Phase II (Table 2). The observed Phase I strategies were used to construct a developmental scale of understanding of how change in one dimension relates to change in another dimension. The observed Phase II justifications gave evidence of a subject's understanding of mathematical generality.

Table 1. Phase I Strategies

Code	Description
Guess	Reported relationship, almost immediately, without describing procedure or supplying justification.
Vis. Boot.	Used a visual model to count the number of units required to fill the space.
Phys. Model	Used a physical model to count the of number of small units required to fill the large space (e.g., poured sand, compared face-sizes, compared string segments).
Exp. Meas.	Measured linear dimensions, computed volumes, areas, or perimeters of small and large solids, then compared those quantities.
Exp. Case	Assigned numerical values to linear dimensions, computed the amounts of sand, paper, or string required for the small and large solids, then compared these amounts.
Alg. Case	Assigned variables to linear dimensions so that a special case of the solid (e.g., a cube) was represented. Used algebraic expressions to represent the amounts of sand, paper, or string required for the small and large solids. Determined the relationship by constructing a ratio of these expressions or factoring the expression for the large solid into the product of a constant and the expression for the small solid.
Alg. Gen.	Assigned variables to linear dimensions so that the general case of the solid was represented. Used algebraic expressions to represent the amounts of sand, paper, or string required for the small and large solids. Determined the relationship by constructing a ratio of these expressions or factoring the expression for the large solid into the product of a constant and the expression for the small solid.
Intrin. Prop.	Used intrinsic properties of volume, area, or perimeter (e.g., dimension, exponent applied to linear measurements) to justify the relationship.
Part/ Whole*	Decomposed the "whole" solid into component "parts," used one of the other strategies to determine the growth factor for one part of the paper or string. Claimed that the growth factor for one part (e.g., small face to large face) would be the same as the growth factor for the entire quantity (sum of all parts) of paper or string (e.g., small solid to large solid). *Part/Whole was always used in combination with other strategies.

Using Sfard's (1991) framework, the Phase I strategies, identified in Table 1, were characterized as process-oriented or object-oriented. When subjects used a process-oriented strategy, they determined the unknown dimension change (growth factor) by a process of comparing measurements.

When subjects used an object-oriented strategy, they determined the unknown dimension change by analyzing the relationship between the known dimension and the unknown dimension, rather than by rotely performing an algorithm. Within the classifications of process-oriented and object-oriented, subcategories were identified that contained strategies based on similar concepts of dimension and change. These subcategories were ordered by level of abstraction to form the developmental scale (Figure 1).

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Table 2. Phase II Justifications

Code	Description
Unjust	Claimed that the relationship was general (or not) without justifying the claim.
Arbitr. Choice	Claimed that since the specific numbers used in the experimental case strategy were chosen arbitrarily, the relationship was general.
Rep. Exper. *	Claimed that repeating one of the non-general Phase I strategies (vis. boot., phys. model, exp. meas., exp. case, or alg. case) provided "evidence" of generality. *Rep. Exper. was always used in combination with Phase I strategies.
Prop. Just.	Claimed that the relationship was general since it was based on proportions. The dimensions of the small shape were unimportant since the relationship was determined by the fact that the large solid's dimensions were twice the small solid's dimensions.
General Vis. Boot.	Claimed that the Vis. Boot. strategy was general for the box problem since the number of small units required to fill the space was independent of specific measurements.
Alg. Just.	Claimed algebraic solutions were general due to the arbitrary nature of variables.
General Intrin. Prop.	Claimed that a solution based on intrinsic properties (e.g., dimension, exponent applied to linear measurements) was general due to the independence of the intrinsic properties from the specific shape.

1. Physical / Visual Process (Phys. Model and Vis. Boot.) 2. Numerical Process (Exp. Meas. and Exp. Case) 3. Algebraic Process (Alg. Case and Alg. Gen.)	} process – oriented
4. Dimension Analysis (Intrin. Prop.)	} object – oriented

Figure 1. Developmental Scale of Process-Oriented and Object-Oriented Strategies

A subject's placement on the scale was determined by the strategy used. The first strategy successfully implemented indicated the subject's independent performance. The strategies used later in the interview indicated the subject's performance under adult guidance. When the least and most sophisticated strategies used by a subject were close on the scale, the subject's Zone of Proximal Development (Vygotsky 1978) was said to be "narrow." When these strategies were far apart on the scale, the subject's Zone of Proximal Development was identified as "wide." The number of prompts and probes used by the interviewer to assist the subject in implementing a strategy successfully was also noted in order to characterize the subject's facility with the strategy.

During the interview, both of the robust solvers of geometric related rate problems advanced to the Intrin. Prop. strategy, from different starting points and requiring different amounts of guidance. Both of the weak solvers began with very visual and hands-on strategies (Exp. Meas. and Vis. Boot.) and required significant amounts of guidance to successfully implement the Alg. Gen. strategy. The outlying subject who scored low on Test 1 and much higher on Test 2, began with an Alg. Case strategy and advanced to a weak Intrin. Prop. strategy. The other outlier who scored high on Test 1 and much lower on Test 2, first successfully implemented an Exp. Case. strategy and never advanced.

Conclusion

In order for subjects to understand processes performed on dimension changes, such as the rates of dimension change in related rate problems, they must first be able to think of those dimension changes as objects. Even the most robust solvers of geometric related rate problems examined in this study, who were able to perform the procedural skills independently and in concert, required adult guidance to advance to object-oriented conceptions of dimension changes between two static states. The weakest solvers of geometric related rate problems were not able to advance beyond the process-oriented approaches, leaving little hope that they could conceptualize dynamically related changes, if they could compute them.

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A STUDY ABOUT THE METHODS USED BY COLLEGE STUDENTS TO MODEL REAL SITUATIONS

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The objective of this research was to find out the types of models and methods that college students use to pose real problems in the sciences. In the study, forty two chemical engineering students worked eight real type problems. Although all problems were of exponential type and typical of a treatment with differential equations, the students used mainly arithmetic procedures and linear models. The resource most employed was the rule of three[1]. This study shows the limited ability of the students to apply the mathematical techniques they have encountered in previous courses and hints on some ways to improve teaching.

Introduction

The construction of mathematical models is an important activity in the sciences. Historically, the development of mathematics was always tied to the resolution of real problems of the era. However, in this century, there has been a separation between mathematics and other sciences, which can be seen reflected in the teaching of all these disciplines.

Mathematical modelling as an approach to teaching mathematics and in general sciences, has gained considerable strength in some countries, for example England (Mellar et al, 1994). With the recent advances in computational tools, especially spreadsheets, this approach appears even more attractive.

Due to the previous arguments, we believed it was important to find out the methods that college students, who have not yet received any previous instruction based on mathematical modelling ideas, would follow in posing and solving problems of real situations. Also we were interested in finding out what kind of mathematical tools and resources they would employ.

Methodology

For this study we designed a questionnaire with eight problems, each one with different contexts and characteristics (the list of problems are listed in the appendix).

The questionnaire was applied to 42 college students of the Chemical Engineering Department of the "Instituto Tecnológico del Estado de Hidalgo" in Mexico. These students were in their fifth semester and had already taken a course in Differential Equations and another complete course in Integral and Differential Calculus. Each student was allowed to take the questionnaire home for a period of two weeks. After this, some of the students were interviewed to clarify their answers and to probe their ideas further.

The analysis of the answers was done as follows. First, the models generated by the students were divided into: 1) *Linear Models* and 2) *Exponential Models*.

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[1] the "rule of three" refers to the rote algorithm of posing the problem with an equality of the form $A/B = x/D$ and then using cross multiplication to solve for x.

Each one of these two groups was further subdivided into four categories according to the mathematical tools used by the students: A) *Arithmetic*: using only numerical calculations, B) *Algebraic*: using variables, C) *Differential Equations*: using derivatives and D) *Difference Equations*: using recursive relationships.

Also, we tried to compile the type of resources, like graphs or tables, that students used as an aid to solve the problems.

Results of the analysis

In the first problem of a human population, the distribution of answers was as follows:

Linear:	35
Arithmetic:	32
Algebraic:	3

Exponential:	6[*]
Arithmetic:	6

The arithmetic linear procedures were in a vast majority (27) a simple application of the rule of three. The other 5 constructed tables for their solution. The 6 exponential models consisted of the calculation of the percentage increase of the population, which was then applied repeatedly.

In the second problem of radioactive disintegration, the results were:

Linear:	39
Arithmetic:	39

Exponential:	0

In all the answers, the students utilised the rule of three to solve this problem (in 100 years 250 gr. disintegrate, thus in 30 years, 75 gr. will disintegrate, having left 425 gr.).

For problem 3 of compound interest, the distribution of answers was the following:

Linear:	7
Arithmetic:	7

Exponential:	32
Arithmetic:	25
Algebraic:	7

The arithmetic linear procedures fell within the rule of three. The arithmetic exponential ones rested all on the use of a table, applying progressively the interest given. The algebraic methods used formulas of compound interest like: $1000(1+0.08)^n$.

It is apparent the strong contrast between the answers to this problem and the previous ones. This problem shows that many students can produce solutions of exponential type.

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[*] *In this and the following pairs of tables, the rest of the 42 students are not reported (in this case: 42 - 35 - 6) didn't answer the problem.*

For problem 4 of cooling of an object, the breakage of the answers was:

Linear:	36
Arithmetic:	30
Algebraic:	6

Exponential:	0

All the arguments given by the students were of the linear type. From the arithmetic procedures, 25 applied the rule of three and 5 constructed a table. The equations of the 6 students that answered with algebraic methods, were all of the form: $T = 210 - 10t$.

The temperature in this real situation should behave like a decreasing exponential, stabilising at the air's temperature. We have to clarify that five minutes of cooling is a relatively short time, and therefore a linear approximation would be also appropriate (we doubt that the reasoning of the students was along these lines).

In the graph requested in this problem, we observed that most students only plotted their numerical data without giving a complete graph. Although there were some interesting graphs (like a straight line decreasing up to the air's temperature, which was continued with an horizontal line), this aspect has to be investigated more in depth.

In problem 5 of the evaporation of a sphere, only 25 students answered:

Linear:	12
Arithmetic:	3
Algebraic:	9

Exponential:	13
Arithmetic:	5
Algebraic:	1
Differential Eq.:	7

For the first time there appeared solutions based on differential equations, taking into account the variation law given in the problem (in problem 2 the variation law was also given but there were no solutions of this type). The equations proposed were all different but a typical one was: $r' = k4\pi r^2$. None of these procedures were developed very far.

The arithmetic methods were based mostly on tables. The algebraic equations used were: $y = 8 - 2x$ for the linear ones and $r = (1-0.25)^t$ for the only exponential.

In problem 6, which is a repetition of problem 1, but blocking arithmetic procedures (by requesting explicitly an equation), almost half the students (19) didn't do any work:

Linear:	21
Algebraic:	21

Exponential:	2
Algebraic:	1
Differential Eq.:	1

We can note that the students insist on linear models. The algebraic equations proposed were of the form: $y = 5 + 0.22t$ for the linear case and $P = Po\ (1+0.22)^t$ for the exponential. The only differential equation was: $P' = \beta P$.

Problem 7 (the same as problem 3, but asking for an equation) was solved by only 13 students. All of them using algebraic procedures. Six linear of the form: $y = 1000 + 80x$, and seven exponentials, using formulas of compound interest as in problem 3.

Problem 8 (a repetition of 4) was solved by only 15 students. From them, 14 applied algebraic procedures: (11) linear and (3) exponential. There was a single differential equation: "$T' = T_{object} - T_{ambient} = 220 - 20$".

In the interviews (with seven of the students) we tried mainly to find out why the linear models were used so frequently. In general the students pointed out that the rule of three is the resource they use to solve the majority of the problems they encounter. For problem 1 of a population, most of the students agreed that the linear model was not the adequate one, but that they didn't know other ways of solving it. However, for problem 2 on disintegration, all of them except one stated that the behaviour should be linear.

Conclusions

The main findings that can be extracted from this research are the following:

- Although the problems were of exponential type, the approaches were mostly linear. The exception was the well known context of compound interest.
- All the problems could be treated using either difference or differential equations, however, the arithmetic procedures were applied by far the most. When an equation was asked explicitly, 96% of the times the students opted for algebraic ones.
- The rule of three was the resource most employed.

We can clearly see that these college students can not apply successfully the mathematical tools already studied by them, to pose and solve problems based on real situations. Not only the techniques of calculus and differential equations were hardly used, also their algebraic language, which is so useful to make generalisations, is somewhat buried in the mathematical arsenal of the students.

We could also notice that although most of these students can use an exponential model to describe the compound interest situation, they can not connect it to other situations. This suggest that mathematical instruction should focus not only on a good understanding of mathematical concepts and techniques but also to facilitate these connections with real situations and between situations themselves.

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Appendix

In what follows, we give the list of problems used in this study, together with their characteristics, according to: A) Subject, B) Continuous or discrete variation, C) Type of behaviour and D) Additional information provided or requested:

Posed Problems:	**Characteristics:**
1) In a country the initial population is 5 million people. After 5 years, the population reached 6.1 millions. Predict the population in 20 years. Draw the graph of the population against time.	A) Human population. B) Continuous variation. C) Exponential growth. D) None.
2) The half-life of a radioactive substance is the time needed to disintegrate to half its mass. A 500 grams radioactive element has a half-life of 100 years. If the rate of disintegration is proportional to the quantity of the element present at the time, predict the quantity of this element after 30 years.	A) Radioactive disintegration. B) Continuous variation. C) Exponential decrease. D) Variation law is given.
3) An initial amount of $1000 is put on a bank at an annual rate of 8%. How much money will it be in the account after three years? How long will it take to double the amount?	A) Compound interest. B) Discrete variation. C) Exponential growth. D) None.
4) An object at 220°C is exposed to a 20°C air temperature. After a minute, its temperature goes down to 210°C. Predict its temperature after five minutes. Draw the graph of its temperature.	A) Cooling of an object. B) Continuous variation. C) Exponential stabilisation. D) None.
5) A naftaline sphere of 8 mm in radius evaporates at a rate proportional to its surface. If after a month, its radius is 6 mm, build a mathematical model describing this decrease and solve it.	A) Evaporation of a sphere. B) Continuous variation. C) Linear decrease (of radius). D) Variation law given.
6) Return to problem 1) to obtain an equation relating the population P with the time t, using the hypothesis that its rate of growth is proportional to the population at the time.	Same as in problem 1), except for: D) Variation law is given and an equation is requested.
7) Return to problem 3) to obtain an equation relating the amount C in the account with the time t.	Same as in problem 3) except for: D) An equation is requested.
8) Return to problem 4) to obtain the equation of the process, using the observation that the rate of cooling should be proportional to the difference of the temperatures of the object and its surroundings.	Same as in problem 4), except for: D) Variation law is given and an equation is requested.

MENTAL CONSTRUCTIONS USED IN UNDERSTANDING THE CHAIN RULE

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The study analyzes the results of investigations aimed at describing students' understanding of the Chain Rule schema. The development of the Chain Rule schema is analyzed with the help of the triad theory of Piaget and Garcia. The triad theory postulates the existence of three stages, intra-, inter- and transoperational stage along which the development of a concept proceeds. We reinterpret the triad theory in the context of the Chain Rule schema, provide the typical examples of students' thinking for each stage and shortly discuss the pedagogical implications of our analysis.

Introduction

This study is an attempt to learn about students' understanding of the Chain Rule concept and its application. Our analysis is guided by a constructivist theoretical framework which we use to describe the constructions as being actions, processes, objects and schemas. Because of the space limitation in this paper we briefly describe only a process of schema construction of the Chain Rule.

Research Framework

There are three main sources that we bring to our systematic approach to describing an advanced mathematical concept and its acquisitions: (i) theoretical analysis; (ii) instructional treatment; (iii) observations of students in the process of trying to understand the concept.

Ad(i): Our purpose here is to propose a model of cognition, called genetical decomposition, which describes specific mental constructions that a learner might make in order to develop her/his understanding of the concept. These mental constructions include actions, processes, objects and schemas. For more information about the research framework and mental constructions mentioned above see Asiala, et al. (in press).

Ad (ii): Following our decomposition of mathematical concepts, we design class and laboratory activities so as to facilitate the students' movement through the hypothesized to be present in the learning of these concepts. Specially designed computer activities performed in a group environment provide opportunities for students to make explicit constructs of the concept in focus.

Ad(iii) The observation of students -- the last phase in our framework—provides us, researchers, with opportunity to gather data, refine the theoretical analysis and design new pedagogical strategies.

Brief review of related literature on Chain Rule

There is a body of qualitative research in students' understanding of functions in various situations (Breidenbach, Dubinsky, Hawks, & Nichols, 1992; Dubinsky & Harel, 1992; Even, 1993; Ferrini-Mundy & Graham, 1994; Lauten, Graham, & Ferrini-Mundy, 1994; Schwingendorf, Hawks, & Beineke, 1992; Vinner & Dreyfus, 1989; and Wilson, 1994). The results suggest that students' notion of a function is substantially different from what they (instructors or researchers) give as a definition. Also, many specific types of functions which represent difficulty to students are identified. Breidenbach et al. (1992), Dubinsky and Harel (1992), and Schwingendorf et al. (1992) suggest that these difficulties indicate the lack of a process conception of function. They offer evidence of an instructional approach which seems to help the students construct this process. Other researchers (Confrey & Smith, 1994; Confrey, Smith, Pilero & Rizzuti, 1991; Cuoco, 1994; Goldenberg, 1988) suggest various computer software and problem-solving techniques to be used in overcoming difficulties with functions. Ayers, Davis, Dubinsky and Lewin (1988) present evidence that computer experiences induce reflective abstraction as students learn about functions and composition of functions. They present a genetic decomposition of composition and probe the process conception of composition in their investigation. Dubinsky (1991) further discussed the issue of composition and reflective abstraction.

Constructing a schema

Within our theoretical framework we define a schema as a coherent collection of processes, objects and other schemas that is invoked to deal with a new mathematical problem situation. Piaget and Garcia (1989) have formulated a triad of stages through which a student passes in her or his cognitive development of a particular concept. These are the intra-, inter-, and transoperational phases of thought. The intraoperational stage is characterized by the student's focus on actions or objects of a transformation in isolation from other actions and objects. Interoperational thought occurs as the student builds relations between these actions and objects through reflective abstraction. Finally, in the transoperational stage these interrelations themselves become the focus of student's reflective activity as they emerge to become the coherent structure of the schema.

Method

Subjects. The 41 students who volunteered to participate in this study were enrolled in engineering, mathematics and science at a large midwestern university and had completed (at least) two semesters of single variable calculus. Of these students, 17 had come from a course which employed computer activities, cooperative learning techniques, and alternatives to lecturing. The remaining 24 students had completed a course taught in a standard lecture/recitation method.

Interviews. Each student was presented with 11 tasks (related to differentiation) by an interviewer and was encouraged to describe her or his solution as the problem was worked. The interviews were audio-taped and later transcribed, and all written work was collected. The transcripts and written work were later coded and searched for issues related to understanding concepts of calculus.

Procedures. In this report we restricted our investigation to the portions of the interviews that deal with the Chain Rule and its application in the context of implicit differentiation, related rates and the Fundamental Theorem of Calculus. In addition, students' written work completed during the interviews were analyzed. Students' understanding of the Chain Rule concept was coded in relation to the Action-Process-Object-Schema (APOS) epistemological framework (Asiala et al., in press). Variability in coding results suggest that Piaget & Garcia's triad of inter- and trans- stages which address the variety of relationships among particular cases, would better explain a development of the Chain Rule schema. In the next section we describe the three stages of the Chain Rule schema development together with the examples from our interviews that support each one of them.

Analysis and Discussion: Constructing a Chain Rule schema.

Below we use a three stage development of schema (intra-, inter-, and transoperational) to describe student's construction of the Chain Rule schema. We say that a student may show intraoperational thought with respect to the Chain Rule when he or she approaches a problem situation by following a set of rules or formulas which are specific to that situation. It is the situation that drives him or her to proceed with certain actions. This may be indicated by laboriously identifying two functions, differentiating each and plugging the results into the formula. The following excerpt from our data illustrate the above situation. When asked to explain how to find the derivative of the function f(x) = (1-4x^3)^2, Dale said: "All right…computing the derivative of one minus four cubed the quantity squared. So, these are the exponent…bring down the two. If you have something to the n minus one which in this case is one and you multiply by the derivative inside." Here the student is focusing on the situation at hand—an intra-phase characteristic—and invokes the Power Rule rather than more general Chain Rule.

The interoperational phase can be seen as the student begins to collect the objects and processes in a set associated with the term "Chain Rule". The student is building the relationships between actions, processes and objects through reflective abstraction. This is well illustrated in the following situation. When asked to find the derivative of the implicitly given function x(y)^2 + y(x)^2 = A (A being a constant) Sarah explains: "Well, there is a lot of ways you can approach this, I think. You could assume that y is a function of x, and then you have to compensate for that, because the chain rule is involved here a bit. If y is a function of x, you could kind of look at this as composition."

Sarah is here in the process of building the relationship between the implicit differentiation and the Chain Rule.

Below is another quote from Dale in which we can observe that he already constructed already the relationship between the Chain Rule and the Power Rule. These rules are treated in previous Intra phase as separated, unrelated entities.

Dale: "The chain rule is with the composition of functions, right? So, if you have f of g of x then, then the derivative of f of g of x would be f prime of g of x times g prime of x. OK. F would be x squared and g of x equals one minus four x cubed. I guess you could make this s and t... You have f of s equals s squared and then you do f prime of s it' equal two s. Then when you go from f of g of x to f prime of g of x you know that f prime of s is two s, so you multiply two primes whatever g of t is which in this case is one minus four t cubed and then f of t and you take g prime of t it's equal to negative twelve t squared. So, you use the chain rule to get f prime of g of x times g prime of x."

Once the collection attains a coherence as a schema the students moved to the transoperational stage. That is she or he is operating on the mental constructions which make up the collection. In other words, she or he is now able to reflect explicitly upon the implicit structure of the Chain Rule. She or he can give some explanations as to why the Chain Rule is true and has a way of deciding when to apply it or not. Jan is reflecting in the situation below upon the chain rule while answering the interviewer question: "What do you look for when you decide to use the chain rule?" Jan said that she looks for"...where the function is, a variable, where it is to the first derivative. If it is not you might have to continue further. Like normally if the function is just x, then we don't use the chain rule. Suppose it is a variation of, like a power of x or an integral multiple of, then we use the chain rule."

After the interviewer comment:" So you look for some sort of form?" Jan replied: "Yes."

Other illustrative examples of the triad stages will be presented at the conference.

Summary

In general, the results of our study suggest that understanding of the Chain Rule presupposes the subject's knowledge of at least, at the process level, a function, composition of functions and differentiation. Variations of coordination between these concepts determine the level development within the triad. In particular, it appears that the scope of the student's function schema plays a decisive role in that development. Thus an important pedagogical suggestion we can offer here is to consciously emphasize and develop the function schema of our students as we acquaint them with different manifestations of the Chain Rule. In general, we need to provide the students with the variety of experiences that would lead them to progress from the intra- stage through inter- to transoperational stage of understanding the Chain Rule concept.

Authors' note: The paper is the report of a much broader study on students' understanding of the Chain Rule whose authors are: Julie Clark, Francisco Cordero, Jim Cottrill, Bronislaw Czarnocha, David DeVries, Denny St. John, Georgia Tolias and Draga Vidacovic.

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STUDENT PERCEPTIONS OF ALGEBRA AND ITS APPLICATIONS

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The National Council of Teachers of Mathematics (1989) emphasized the importance of school mathematics programs instilling in students a "valuing of the application of mathematics to situations arising in other disciplines" (NCTM, p. 233). In this study, an eighth grade algebra class (N = 27) completed a written questionnaire on their perceptions of the usefulness of mathematics in other areas of the curriculum. While students felt strongly that mathematics was important in the hard sciences, they saw very little use for mathematics in social studies and even less in Language Arts.

The students were then taught a unit from the area of discrete mathematics which included a study of genetics and its connection to probability, as well as an exploration of voting and election theory and its connection to functions and counting. Throughout the 17-day (4 school weeks) unit, student reactions and comments in the classroom were recorded. Also, students were given daily journal reactions to write, and comments from the classroom and journals were analyzed for patterns in student responses. At the end of the unit, students responded to an open-ended survey and engaged in informal interviews.

The following conclusions were drawn through the comparison of pre-unit surveys and evidence of attitudinal changes, as measured by observations, post-unit surveys, and interviews: (1) Student perceptions about the use of mathematics in social studies changed significantly due to the incorporation of voting theory in the discrete mathematics unit. (2) The introduction of applied topics sparked what the literature refers to as "continuing motivation" in many students--the desire to pursue more knowledge in the area, beyond that which is required in the classroom. (3) The most common word used by students to describe the discrete unit was "interesting," and they generally attributed this to applicability and the lack of a single answer or procedure to solve the problems. (4) The unit appeared to reinforce several algebraic concepts that students had developed earlier in the school year and served as a context that demands their mastery (e.g., exponents, graphing, patterns, recursive thinking). This context also appeared to engender confidence among the students. (5) Direct instruction on mathematical process skills and the components that make up algebra were instrumental in helping students to appreciate its usefulness.

This study illustrated the importance of incorporating "real life" problems in the algebra classroom to help students appreciate the applicability of the content area. Discrete mathematics topics were particularly useful in that they served as fertile ground for exploring practical applications of mathematics.

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RESEARCH ON TEACHING CALCULUS: THE ROLE OF WRITING AND COOPERATIVE GROUP LEARNING IN DEVELOPING CONCEPTUAL UNDERSTANDING IN CALCULUS

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In far too many classrooms, students who study calculus are either unsuccessful or resign themselves to learning strategies to cope without understanding. Two educators critically examine the role of writing and cooperative group learning (CGL) in developing conceptual understanding in calculus. Leslie Aspinwall, as teacher-researcher, taught a section of first-semester calculus, and Diane Miller served as his critical friend in an action research project. The study investigated how the pedagogy of Calculus I can be changed to help increase students' *conceptual* understanding of calculus. Specifically, the research questioned whether the use of writing and cooperative group learning would (1) improve students' attitudes toward learning calculus, and (2) help students construct conceptual understanding of Calculus I concepts.

In order to effect a reorganization in calculus to learning groups and writing, two pedagogical changes were implemented during this research project. First, CGL was implemented throughout the semester to introduce student-centered instructional strategies into the course. Second, at the end of each GCL activity, a writing activity was used to hold individuals accountable for learning and to provide students with the opportunity to communicate their understanding in writing.

The study was organized into three components: (1) data collection, (2) data analysis, and (3) curriculum development. Data sets collected from the experimental section of Calculus I included 26 student journals, 2 researcher journals, 18 cooperative learning group activities, 25 writing prompts, and 5 student interviews.

Assertions resulting from the analysis of various data sets suggest: (1) Students perceived the act of writing as promoting conceptual understanding; (2) Students perceived GCL as a worthwhile strategy for promoting conceptual understanding; and (3) Students enrolled in Calculus I initially resisted the use of writing in class, but by midterm, this negative attitude had shifted to one of positive reliance; that is, they asked to continue the writings past mid-semester.

THE ROLE OF THE VISUAL AND ANALYTIC ACTS IN THE STUDENTS' UNDERSTANDING OF LINEAR DIFFERENTIAL EQUATIONS

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This is a research project to study the relationship between visual thinking and analytic thinking in student's strategies in solving mathematical problems. We will use calculator with graphic and programmable capabilities. From the function and their graphs to graphical solution of linear first order differential equations we will study the relationship that exists between visual thinking and analytic thinking in a student's strategies when she or he is solving a problem. The aim of this study is to understand how a student deals with visual thinking and/or analytic thinking when she or he tries to respond to a problem. We assume that visualization and analysis are mutually dependent in mathematical problem solving, rather than a dichotomy. To study these relationships is important just for this dependency. In our study we are using a model that assumes this dependence between analysis and visualization (Zazkis et al 1996).

A sequence of activities with a graphic calculator was designed to explore the student' strategies (Cordero & Solís 1995). The sequence consists of four activities. We considered a specific function and evidence it throughout different situations. The following element play an important role analysing them through mental construction (Asiala et al 1996) of students: a) Control procedures, procedures applied to graphs, these consist to move them and find patterns of behavior in the expression $Y=A[f(x)+B]$, viceversa, when A and B are modified, patterns in the graph of a function are found; b) Variation of coefficient, the coefficient as the variables, that is the expression $Y=A[f(x)+B]$ is substituted by $Y(A, B)$; and c) Global situation, the function is an object, in the sense of mental constructions, i. e. the whole curve is an object, a process is not perceived, thus the function and its graph are the object to operate.

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HOW DO I KNOW WHEN MY SOLUTION IS CORRECT? STUDENTS' BEHAVIORS IN AN ADVANCED CALCULUS COURSE INVOLVING A COMPUTER ALGEBRA SYSTEM

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This study explores upper-division college students' mathematical understanding in a course, Advanced Calculus for Engineers and Scientists II, in which a computer algebra system (Maple) was used. The data for the study were collected through cycles of observation/interview sessions with students as they attempted challenging sets of homework problems which counted as 35% of the final grade.

The aspects of this data to be discussed in this presentation concern the: a) criteria, knowledge, heuristics, and strategies students used to decide whether they have arrived at valid solutions to mathematics problems? b) ways in which approaches used by a group of students were the same or different from those used by students who initiated work on the problems alone? c) modes of *students'* uses of a computer algebra system to solve problems or confirm that their solution methods were correct.

The first half of the course (seven weeks) was the focus of this study. That part covered functions of a complex variable, some of Cauchy's results, applications to fluid flow, contained four of the ten-problem homework assignments, and ended with the mid-term examination. Five of the fifteen students enrolled in the course agreed to participate in the study (two men and three women). All five students were physics majors. Two of these students preferred to do their initial homework problems alone. The other three worked together as a group throughout both this course and its prerequisite.

This study is based primarily on qualitative research methods in the spirit of holistic, naturalistic-inductive inquiry (Patton, 1990). It makes use of aspects of Schoenfeld's problem solving framework (Schoenfeld, 1985). The data include: audiotapes and filed notes for homework sessions, audio and videotapes of a "Topics Interview" (which explored understanding of prerequisite concepts such as function and inverse function), copies of students' tests and written work, their Maple worksheets, class notes, the professors Maple worksheets, and interviews with the instructor.

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THE MEASUREMENT OF ABILITIES TO LEARN PROCEDURAL AND CONCEPTUAL KNOWLEDGE FOR THE PREDICTION OF PERFORMANCE IN COLLEGE CALCULUS

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The purpose of the study was to provide measures of students' abilities to learn procedural and conceptual mathematical material, and to determine the extent to which these abilities are related with each other and with students' abilities to master standard mathematical exercises in first-year college calculus. A test instrument was developed consisting of four subscales to measure abilities for procedural acquisition (P1), procedural application (P2), understanding mathematical symbols and operators (C1), and understanding conceptual rationales for mathematical procedures (C2).

A sample of thirty-three first-year college calculus students was used in the test development process, and an additional sample of forty-four students drawn from four intact first-year college calculus classes provided the sample for examining the inter-relationships of the measures and their correlations with students' performance on routine exercises on their college calculus final examinations.

The overall test reliability was alpha=.84; the subscale reliabilities were: P1, alpha=.48; P2, alpha=.65; C1, alpha=.65; C2, alpha=.51. Interrater reliability was .94. Correlations between P1 and P2 were relatively high (r=.57); correlations between C1 and C2 were .48. C1 was highly correlated with both P1 (r=.73) and P2 (r=.74). Correlations of C2 with P1 and P2 were considerably lower (r=.40 and r=.39, respectively). Correlations of procedural and conceptual abilities with performance on calculus exercises in order of magnitude were C1 (r=.42), P2 (r=.40), P1 (r=.34), and C2 (r=.10).

Other variables that correlated significantly with calculus performance were prior high school mathematics grades as well as Mathematics and Verbal SAT scores. Students' strategies studying for their final examination did not correlate positively with calculus performance.

The most predictive models of calculus performance that included abilities to learn procedural or conceptual knowledge were: (1) the combined procedural subtests with grades in third-year high school mathematics; and (2) C1 with grades in third-year high school mathematics. Each of these models accounted for thirty-six percent of the variance in calculus performance.

ALGEBRAIC THINKING

ALGEBRA IN MOTION: STUDENT CONCEPTIONS IN A CONTEXT-RICH MIDDLE SCHOOL PREALGEBRA CURRICULUM

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In January, 1996 our research group began an eight-week curriculum project introducing the entire seventh grade class of one middle school (n=250) to algebra. Fifteen seventh graders were interviewed at the end of the school year in order to examine their functional thinking. This paper studies three cognitive issues that arose during the interviews. First, the role of context is highlighted as students moved easily between the hiker context included in the curriculum and a context involving pay scales which was not part of the curriculum. Second, evidence of ratio based thinking and support of Confrey's splitting conjecture (Confrey, 1988) are discussed. Finally, the interviews reveal student strategies in distinguishing between variables and their associated units.

I. Overview of Curriculum

Our research group worked with both of the seventh grade teachers in order to develop a curriculum which would be responsive to their concerns about content coverage. As such, the curriculum involved functional reasoning and algebraic strategies up through simultaneous equations. Students were introduced to the curriculum by way of a distance-versus-time problem involving hikers, a context which was periodically revisited. Using motion detectors and a software package, MacMotion, students analytically examined qualitative graphs of data created by their own motion. Then, students gained familiarity with Function Probe (Confrey, 1991-1996), software that invites exploration of functional relations. Following these activities, they began exploring Cartesian graphing. Multiple forms of notation were integral to our project as students quantified various data using tables, graphs and equations and developed proficiency in identifying patterns within these representations.

II. Interview Details

In order to gain insight into individual students' functional thinking, paid interviews were conducted with 15 students. The same curriculum was offered to the entire seventh grade, which consisted of "expanded" and "regular" classes taught by both teachers. The ratio of students in "expanded" classes to those in "regular" classes was 3:2. To make the interviewees representative of the seventh graders, we used a random number drawing without replacement to select 9 students from the "expanded" classes and 6 students from the "regular" classes.

Each student met with one of the authors for a 30-40 minute videotaped interview where blank paper, graph paper, a calculator, and a computer with Function Probe software were available. The question asked the student to compare the following three babysitting pay scales:

A. $2.00 per hour
B. $4.00 for showing up, then $1.00 per hour
C. No pay for the first 2 hours, then $3.00 per hour after that.

The problem then asked the student to "make a guide in the form of a chart or graph or formula to put by the phone to allow you to quickly decide which pay scale to use."

III. Changing Context from Curriculum to Interview Problem

In using monetary rates as the basis for the interview question, we deliberately altered the context from a motion-based one. Nonetheless, many examples from the interviews show that the students drew on the familiar hiker context to frame the new problem they faced. Todd's example is the most blatant: he calls the payment plans "guys" and labels the amount of money received "position."

> Interviewer: When you say position, what do you mean by position?
> Todd: I mean his like place, where he, how much money he's earned at a certain time, like the place he's at.

When asked what her favorite aspect of the early part of the curriculum was, Rebecca responded, "I guess the motion detectors when you had to like walk a straight line, because it was sort of a challenge to do, and you got to find out the best way to walk for each line to be straight or curvy or something." This experience notably influenced Rebecca's conception of the interview question, as revealed by her description of straightness in the two contexts:

> Interviewer: What about the speed makes...the line go straight?
> Rebecca: Well, if the speed is like even, and if it goes like one mile per hour, instead of like one mile and then four miles in an hour and then like one mile again, then it will make it like a little more crooked.
> Interviewer: ...what makes these lines straight, in the babysitting problem?...
> Rebecca: If for each hour, you have like a steady rate of money that is being charged, instead of like 4 dollars for one hour and then like 2 dollars.

These excerpts suggest that giving students a context in which to develop ratio reasoning constructively informs their conceptualization of similar problems in new contexts. Rather than restrictive, the original context helped these students to conceive of mathematics as a tool they can use to analyze a new situation.

IV. Ratio-Based Thinking and the Splitting Conjecture

Our interviewees frequently exhibited ratio-based thinking when using equations, graphs, and especially tables. Rebecca, for example, explained how she came up with the equation for plan B as "$y = 1x + 4$" by contrasting the 4's actual status as a constant with its potential status as a *rate*, saying, "it's... one dollar per hour, and you get 4 dollars just for showing up... so you would just add on 4 dollars, cause it's not 4 dollars per hour." With graphs, most students used the phrase "one up, one over" to indicate the slope of a line, a concept which involves two covarying entities.

In working through Option C, Heather gives us insight into her tabular covariational thinking. She began by creating the following table:

h	$
1	3
2	
3	
4	
5	
6	12

Inspired by the 3 and the 12, she then entered "6" and "9" into the "$" column, but evenly spaced vertically, rather than across from a corresponding "h" value. Seemingly confused about the different number of entries in each column (6 in the left and 4 in the right), she verbalizes her thinking: "Hold on a minute. What? Oh... my table. I think I messed up. It can't go by threes. Because 3 times four is 12. Not 3 times six is 12." Here, we see evidence of Heather using first an additive (left column) and then a splitting (right column) scheme. Having split 12 into four groups of three, Heather then compared the number of groups in each column. She demonstrated a desire to have the same number of groups in each column in order to create a covarying relationship between hours and dollars. After rereading Option C, Heather achieved this by putting zeros in for the first two dollar entries, and then placed 3, 6, 9, and 12 across from their hour counterparts. Heather's activities are congruent with Confrey's observations (1995, p. 292) of children's proficiency with additive and multiplicative strategies and one-to-one correspondence.

V. Student Strategies in Distinguishing Between Variables and their Associated Units

We have chosen to examine Option C ("No pay for the first 2 hours, then $3.00 per hour after that") as a case study because it was invariably the most challenging plan for each of our students to model. Its elusiveness lay in the fact that the equation we sought, $y=3x-6$, has as its added constant a number not directly present in the wording of the plan. Because students were not familiar with the parenthetical notation $3(t-2)$, where the numbers in the plan are clearly shown, it is important to recognize the role of students' understandings of units as they derived the equation. Some of our previous test data highlighted the difficulty of this type of problem; when a question of this sort was posed in the hiker context, 61.5% of the 130 who took that test missed only the added term of the equation, and 76.3% of that group used the time constant (here, 2) as their added constant.

The diverse paths which our interviewees pursued to come up with equations for plan C give us fascinating insight into (a) the individuality of each learner's constructed knowledge, and (b) the agility with which students of this age move productively, and often spontaneously, among representations. To highlight the diversity of preferences present in our small cohort sample, we give here several examples of students working toward the same end within a verbal, a tabular, or a graphical representation.

(a) verbal. Three of the students got -6 directly from the wording of the problem. Only one of these, Alison, kept this as the added term of her equation. When asked where she got it, she replied simply "3 times 2." That she thought of this as money is indicated by the fact that she revised her first attempt at the equation, y=3x-2, to y=3x-6 because the 2 was "in hours." Lisa came up with the equation y=3x-6 from the plan's wording in the first few minutes of the interview, saying "minus 6, because 2 hours of $3." However, Lisa later decided that this equation was wrong because negative pay for the first two hours made no sense to her. Heather, who never had -6 in her equation, said of C, "I subtracted 2 hours 'cause there was no pay... and that was 6... I had to subtract by 6."

(b) tabular. Several of the interviewees favored this approach and used tables extensively, especially to compare plans. At least three achieved key insights into C's equation through tables. Trevor employed tables in two ways, first using a ratio notion combined with trial-and-error:

> Hmm. I'm not sure about this C one. Wait a second. I'll think about the delta. This goes up 3 every time [makes a mark next to C column], this goes up 1 [makes a mark next to hours (x) column], so it's 3 over 1, um,... 3x [pause] minus 6. [pause] Yup. 3x-6. And I got that by thinking, the slope was 3 over 1, right? So I do C equals... 3x. So I take that and just use an example. 3x3 = 9, and that's not right. So minus 6 is 3. So it works for that, and I tried the other ones.

Next he offered an alternative in which one table is shifted to match another.

> Trevor: C, adjusted. [Writes a new table beginning with 3 rather than with two zeroes.] 3, 6, 9, 12, 15, 18, etc. So 6 would be where the second zero is [in the first table for C]... You have to take 6 away to make it [the second entry] zero, so that this [the original table] is correct.
> Interviewer: ...Are you saying that this [the new table] would be the regular 3x?
> Trevor: Right, that would be 3x. This [the original C table] is 3x-6, because you take 6 away from each one of 'em.

(c) graphical. Because of the monetary context of the problem, most interviewees considered the graph of plan C to have only nonnegative values. The connotation of a negative y-intercept thus understandably kept them from using -6 in their equations. Most focused on the x-intercept of C's line more than on its y-intercept. Students who viewed the problem graphically were able to modify their equations for C by considering the units represented by each axis. Kate, who came up with 3x-2 from the wording of the plan, kept the axes in mind when asked to show where the -2 in her equation appeared on her graph:

> Kate: This one would be the negative two [she indicates the point (2,0), the x-intercept of her line]... But, this [the x-axis] is in hours, so it's negative two in *times*.
> Interviewer: So on this part [y=3x-2] you're saying... the 2 represents time here?

> Kate: Yeah,... this is minus 2 hours. Or, minus 2. [Looks at paper, pauses, raises eyebrows in confusion.]
> Interviewer: So did you say, minus 2 dollars?
> Kate: Yeah, kinda. Minus 2 *units* of a dollar, like, they're in threes.

Like Kate, Alan begins with y=3x-2, but later comes up with a quintessentially graphical explanation for y=3x-6:

> You can't subtract hours from here [indicates the y-axis with his mouse], 'cause this is talking about how much money they make. So you have to subtract the money. Which would be 2 times 3.

VI. Conclusion

Our research has revealed that, with an appropriate curriculum and with greater ease than is generally expected, seventh graders can meet the recommendations of NCTM's *Curriculum and Evaluation Standards for School Mathematics* for patterns, functions, and algebra in grades 5-8 (pp. 102-104) and demonstrate the first of the corresponding skills in grades 9-12: "represent situations that involve variable quantities with expressions and equations; use tables and graphs as tools to interpret expressions and equations" (p. 150). We hope that this study can promote the inclusion of algebra in middle school mathematics which has traditionally been devoted to arithmetic, and contribute to reform in the teaching of algebra for all students.

Taking a look at the differing thought processes these students have used in constructing an algebraic equation from the same contextual problem helps us to better appreciate the fruitfulness of encouraging learners to maneuver among various perspectives and means of representation rather than focusing on traditional symbol manipulation. In asking students to come up with an equation, this case study represents, in microcosm, middle school students' cognitive transition into symbolic algebra. Our research has shown that seventh graders can make this transition when given contextual problems and encouraged to use the various sorts of numerical reasoning skills with which they are already familiar.

Knowledge of what constitutes an effective transition to algebraic thinking is imperative in the current reform climate. Our research, which underscores students' abilities to develop robust algebraic thinking through a functions-based approach, helps to meet the demand that challenging content be taught to all students. This research calls for a precise articulation of skills and concepts which support student success.

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PROBLEM SOLVING STRATEGIES OF STUDENTS WITH LEARNING DISABILITIES IN PROCESS ALGEBRA

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This study documents strengths and difficulties of an algebra student with a learning disability from five in-depth interviews. The student solved problems in the Hawaii Algebra Process Approach curriculum which develops concepts within problem solving contexts and encourages multiple strategies. The student was successful with an intuitive approach to algebra involving a combination of mental arithmetic, guess-and-test, pattern recognition and his own reasoning. Visual representations of concepts in the form of tables or graphs aided learning. The student had great difficulty with symbolic manipulation of equations and evaluating expressions with a chain of operations. Results suggest that algebra curricula which allow varied approaches and acknowledge student reasoning can greatly benefit students with learning disabilities.

First year algebra has become a "critical filter" (Sells, 1978), in which course success leads to career opportunities in a technological society while failure greatly limits further education and the development of mathematical power (Edwards, 1990). The NCTM *Standards* (NCTM, 1989) recommends that algebra be taught in a constructivist approach with students developing an understanding of concepts through problem solving. However, the special education community has been critical of the reform movement for not addressing the needs of exceptional students. In order to realize a goal of "Algebra for Everyone," mathematics educators need to attend to appropriate accommodations for students with learning disabilities (Parmar, 1995).

Background

In the past, structured approaches and changes within the classroom environment to meet visual and auditory deficits have been recommended for developing mathematical competencies for students with learning disabilities (Burton & Myers, 1987; Sears, 1986). Teacher directed methods such as the use of relevant practice, feedback and reinforcement, strategy instruction, and computer assisted instruction have been shown to be effective (Mastropieri, Scruggs, & Shiah, 1991). The appropriateness of teacher directed methods may be linked to skill development rather than to the development of concepts, problem solving and reasoning. Students with learning disabilities do not necessarily have deficits in their ability to reason mathematically (Bell Mick & Sinicrope, 1983; Miles, 1992; Steeves, 1983). A study by Montague and Applegate (1993) on the development of problem solving skills with middle grades students could detect no significant difference between students with average abilities and those with learning disabilities with the number of cognitive and metacognitive verbalizations. However, studies have also shown that there are qualitative differences between LD students and their average and gifted counterparts. Cognitive deficits have been found with strategies with problem representation for students with learning disabilities (Montague, 1995). In a reform algebra classroom, what accommodations are needed to meet the special needs of students with learning disabilities?

This study takes an exploratory approach employing a teaching experiment to investigate strengths and difficulties of a student with learning disabilities while solving algebra problems. The pedagogical methods exposited in the Hawaii Algebra Process Approach (Rachlin, Matsumoto, & Wada, 1992) directed the design of the project. In this method of instruction, classroom structure involves student discourse, multiple strategies to solve problems, and investigation of concepts through problem sets. The direct teaching of algorithms for students to follow is minimized, with students practicing skills within many contexts. Because of the role of student discourse and the development of concepts over time, students read, hear, and use mathematical vocabulary naturally. This use of vocabulary, not a tenet of the process approach but a product of the approach, may in itself aid retention and development of connections among concepts (Miller & Smith, 1994).

Methodology

Setting

Steve, a tenth grader in a rural high school in the southeast, was repeating a two hour semester class which included half of algebra I. He had an IQ of 113, yet had not passed the state's mathematics competency test for eighth graders. He was receiving special services for learning disabilities in mathematics, reading, and written expression. His algebra teacher described him as a "good reasoner" and felt that he could complete the course if, "I could just get him to put things on paper."

In Steve's high school, algebra I was a high school graduation requirement. With a learning disability in mathematics, Steve could be exempted from the requirement although he still needed this class for credits. The mathematics department in the school was very concerned about students such as Steve and provided varied scheduling and pacing for completing algebra. In addition, all of the algebra teachers were participating in a project funded by the state systemic change grant to redirect instruction in algebra to meet mathematics reform guidelines. During the summer, the teachers and the authors attended a two week class to learn the process approach to algebra. Teachers were now implementing the new methods in their classes and using the text, *Algebra I, A Process Algebra* (Rachlin, Matsumoto, & Wada, 1992). Steve's class covered the first five chapters in the text dealing with problem solving strategies, operations with integers, the language of algebra, solving equations and inequalities, and graphing two variable equations.

Design

The teaching experiment involved five 50 minute audio-taped interview sessions with the student. Each session was conducted in a conference room during one half of the student's algebra class and pertained to the concepts that the class was studying. With the exception of the first interview, the instructor (first author) chose problems for the student to work and discuss from the problem set for homework. With the first session, the instructor began with the problem set the class had been discussing that day rather than the new problem set in order to establish a more comfortable setting. The problem sets in the process approach to algebra are based upon distributed practice of concepts over time, with only one problem in the assignment relating to the new material. As a result, the concepts in the interview sessions were primarily continued practice and building of concepts already introduced. Steve was frequently absent from school; so at times the review topics were new to him. During the five interview sessions, Steve worked fifty-seven algebra problems.

Analysis

The tape transcriptions, work products, and field notes were first analyzed independently by the two authors. The authors listened to the tapes comparing written transcriptions to the original tape to make corrections. Each author then recorded notes of methods the student used, how the student reasoned, what the student could easily do, and what the student had difficulty with. They discussed similarities and differences in their findings to determine particular relationships and themes. One author made a matrix categorizing from each problem, the concepts and skills that Steve was able to do with ease and ones that were more difficult for him. The categories that emerged were: concept of equation, visual representations, tables to equations, solving verbal problems, concept of slope, symbolic form, computational skills, other interesting findings.

Results

Intuitive Approaches to Algebra

Steve was successful when he used a combination of mental arithmetic, guess-and-test, pattern recognition and his own reasoning. In the first session, when he was asked which homework problem was easiest for him, he selected a question which asked, "solve these equations for x." He explained how he found the solution to $5x + 15 = 5$ by reasoning, "I had to get 10, a negative 10 because the answer was 5." In his solution to $5(x+3)=5$, recalling that the answer would be the same number, he said, "...had to be times 1 to get 5." He further demonstrated his understanding of the concept of equation as he explained that for $5(x+3) = 5x + 15$, any number would work. He continued, "because of either side, by one [replacing x with 1] then times five which will be twenty and the same thing here which is two [when x is replaced with 2] and it will be 25 and 25.

In session 2, Steve was asked to work the word problem, "A plumber charges $50 for the first hour of work and $35 for each additional hour or part of an hour. How many hours could the plumber work for $260?" Without hesitation, he subtracted $260 - 50 = 210$. Rather than dividing by 35, he completed the multiplication problem, $35 \times 5 = 175$ then tried, $35 \times 6 = 210$. He then reasoned that the answer would be 7 as he needed to add one more hour. In another session, with the x-y intercept method for graphing linear equations, although he was shown how to substitute the value of zero for x or y on paper, he preferred to calculate the intercepts for $-3x + 2y = 12$ and $2x - 6y = 18$ mentally.

Although Steve was also diagnosed with a learning disability in reading, he did very well with translating verbal expressions into algebraic symbols. In the problem: "I am thinking of a number. If I multiply the sum of my number and 6 by 3, and then add 9, the result is 60", he wrote $(x+6)3 +9 =60$ and obtained the solution applying a combination of intuitive reasoning and guess-and-test. (The instructor reminded him to use parenthesis.)

Steve had great difficulty with using transformation to solve equations and with evaluating expressions containing multiple operations or grouping symbols. When he was asked to solve $3m+5 = 7$, he first tried to think out the solution, then said, "I don't know". When the instructor demonstrated an example of how to manipulate the equation, Steve had difficulty completing the set of steps. With directions such as: "complete each ordered pair to form a solution to the equation, $y = 1/3\ x - 3$; (6, ___), (-9, ___), (0, ____)," he had difficulty substituting in a value for x, then simplifying the expression.

Visual Representations of Concepts

Steve easily was able to interpret functions when graphed on a coordinate plane. From a graph in the first quadrant and second quadrant of the equation $y = -2/3\ x + 5$, he was asked to estimate the value of y when $x = 2$. He quickly responded, "so I just go up from 2. So I go up 4." The instructor responded, "I would ...say... 3 1/2, see how it is not quite 4." Steve then attended to the detail with a better estimate of 3 3/4. With the same graph, he had no difficulty with estimating a value for x when given y or with extending the line to estimate the y value for a point on the graph in the fourth quadrant but not pictured . When Steve was asked to "use the equation to check the accuracy of your estimate," he faltered. Throughout the interviews, in attempting to complete paper-pencil calculations with multiple steps or fractions, Steve had difficulty.

As Steve was introduced to the concept of slope visually as rise/run, a problem asked "draw a line that passes through the point (4, -1) and has a slope of 0. He responded, "slope of 0, wouldn't it just go right here? [motions a horizontal line]." The instructor asked, "how did you know what the slope of 0 would look like, did you just remember that from class?" Steve said, "no, I knew it was flat because the slope of 0 couldn't be any angle." With another problem pertaining to slope, the instructor had Steve graph on a graphics calculator, $y = 2x + 3$, $y = 2x + 5$ and $y = 2x - 2$ to investigate a pattern for writing equations for lines. The instructor asked, "what if I said write an equation which would be parallel (to these) but would intersect at -8." Steve immediately said, "just write $y = 2x - 8$." Steve's greatest difficulty with graphing an equation using the slope and y-intercept involved the direction of the rise or run with a negative number.

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Steve was quite successful with the use of tables to represent algebra concepts. In a problem that he chose as most difficult in one assignment, he made a table to illustrate output from a computer program to evaluate $3x-5x+4$ using $x = -3$ to 3. He was able to explain from the table what the solution to $3x-5x+4=8$ would be. He identified from the pattern he observed in his table, a mistake in his first answer and corrected his computation. When given a problem with a verbal rule "y is half of x" and asked to complete a table, he not only could reason answers using values for either x or y, such as when $x = 100$ or $y = 62.5$, but he could complete the table when m and $x+5$ were substituted for x. Steve was asked to write an equation for this rule. He wrote, "$x = 2y$" from his table rather following the verbal expression. In two other problems with a verbal relationship given and the use of tables, Steve easily wrote symbolic equations.

Conclusions

For teachers of students with learning disabilities, it is necessary to reconsider what are the important goals of algebra in order to provide appropriate instruction. Clearly, Steve would have great difficulty meeting a goal of symbolic manipulation of equations and expressions. Communication was further hindered by his written work in which he had to be encouraged to even write a minimal amount of symbols. Yet, with the process approach to algebra, he demonstrated understanding with many concepts. He was able to learn through his own strengths of intuitive reasoning and visualization of abstract concepts.

With Steve's ability to use tables and graphs to define relationships and solve problems and with his difficulties with symbol manipulation, the use of graphics calculators and other technologies to visualize concepts suggest promise for students with learning disabilities. For example, the difficulty of using symbolic manipulation to solve equations could be replaced with graphic approaches to solutions of equations. The use of table generators could also be valuable for interpreting complex algebra expressions or solving applications.

How typical is Steve's algebra performance of other students with learning disabilities? This can only be answered from further research with larger samples. However, acknowledging in algebra the reasoning abilities of students with learning disabilities and supporting alternative approaches to problem solving will enable these students to develop mathematical power and adapt to a technological society.

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INSIGHT INTO OPERATION AND VARIABLE FROM STUDENT EXPLANATIONS

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In this study we examined preservice elementary teachers' and precalculus students' understandings of unit, variable and operation. This was done by analyzing the students' placements and explanations of points on a number line. There were primarily two approaches used by students in completing this task: (a) estimate, calculate and then estimate, or (b) using additive and multiplicative strategies. We also noted what appear to be several significant misconceptions that only became evident from the qualitative analysis of the students' explanations.

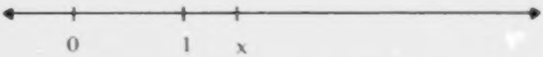
Introduction

Student understandings of operations and of the concept of variable are crucial foundations for building an understanding of the function concept. In this study, we examined the conceptions of two groups of college students on the familiar operations of multiplication, powers and exponents and on their representations on the number line. The students were asked to explain their strategies for carrying out the operations as well as locating their results on the number line. The qualitative analysis of the student explanations revealed weaknesses in the posed task as well as difficulties encountered in moving between numerical and graphical representations of operations and variables.

Theoretical Background

In a study of the relationship between students' understanding of operations and success in beginning calculus, Pence (1995) found that students who dropped out or failed their first semester calculus course had incomplete understandings of the operation of multiplication. In her work, Pence posits the role of the understanding of operations as the cognitive root (Tall, 1992) for the development of an advanced understanding of processes on functions. The following question was posed to beginning calculus students:

> On the following number line, you will see the points representing 0, 1, and x indicated. Approximate the location of the points corresponding to 2, 2x, x^2 and 2^x.



This task required the students to coordinate symbolic, numeric, and graphical representations of quantities, while bringing together units, variable and operation. The variable x was placed at approximately the square root of 2. Pence's results suggest that many students had difficulty in extending their model of multiplication beyond repeated addition with constants. She further posed the question of why so many students were unable to move beyond processing the locations of 2 and 2x. Therefore, in our study, we made two significant extensions in order to better understand students' conceptions of unit, operation and variable. First, we selected both a group of pre-calculus students as well as a group of students with lesser mathematical preparation. Second, we modified the question to ask the students to give written explanations of how they found their answers.

Methodology

Near the end of the semester, we administered the modified version of the question to two groups of students at a moderate-sized, private research university. The first group consisted of 32 students enrolled in a two-semester mathematical foundations through problem solving course. These students were prospective elementary school teachers. The second group consisted of 26 students enrolled in a pre-calculus class. For both groups, we collected the students' final examination score and their final course grade. The modified question was administered at or near the end of the semester for both groups. Hence, we would consider the pre-calculus group of students to be comparable to the beginning calculus students in Pence's study. Unlike the students in that study, these students had access to graphing calculators. The question was further modified to include the phase "as accurately as possible" and administered to a third group of students at the pre-calculus level.

The data collected were analyzed both quantitatively and qualitatively. Each researcher independently scored all of the student responses into the six categories developed by Pence:

(1) unable to locate any of the four quantities correctly;
(2) located only 2;
(3) located only 2 and 2x; (4) located 2, 2x, and x^2, but not 2^x;
(5) located 2, 2x, and 2^x, but not x^2;
(6) located all four quantities correctly.

Both coders found two categories of student responses not reported by Pence:

(7) located 2 and x^2, but not 2x and 2^x; and
(8) located 2, x^2, 2^x, but not 2x.

There was 97% agreement between the coders in their ratings. The Spearman rank correlations between the students' response category and their final course grades and final examination grades were computed for the overall group and for each separate group. The students' explanations were analyzed to discern patterns of student responses and strategies for determining the location of the points. The explanations revealed the extent to which students employed both additive and multiplicative strategies and provided insight into some misconceptions about unit, operation and variable.

Results

The quantitative results of the student responses to the question are summarized in Table 1 for both the mathematical foundations students (Group 1) and the pre-calculus students (Group 2). Note the rather large number (32%) of the students who could only locate 2 on the number line, but not 2x, x^2, or 2^x. While smaller than the 45% found by Pence in her sample, it is nonetheless disturbing that such a large number of students were unable to locate 2x on the number line. In addition, we found 5 students who did not fall into any of the original classifications. Three of these students located 2 and x^2, but not 2x or 2^x. In terms of the hierarchy of responses, we considered this group to be an equivalent classification to category 3. Two students located 2, x^2, and 2^x correctly, but not 2x. We considered this group to be a parallel classification to categories 4 and 5, which are equivalent in the original hierarchy. In contrast to the results reported by Pence, we found that there was little correlation between success on the question and either final course grade (0.124) or final exam grade (0.187) for the pre-calculus students.

category	Group 1	Group 2	total
1	2 6.7%	1 4.4%	3 5.7%
2	10 33.3%	7 30.4%	17 32.1%
3	8 26.7%	3 13.0%	11 20.8%
4	4 13.3%	1 4.4%	5 9.4%
5	1 3.3%	1 4.4%	2 3.8%
6	5 16.7%	10 43.5%	15 28.3%

Table 1. Distribution of Answers by Category by Group

Without analyzing the students explanations, it was frequently difficult to interpret the meaning of the students' responses. Errors in estimation appear to be the cause of many incorrect answers. The differences between 1.3, 1.4, and 1.5 (all estimates that the students gave for the location of x) are great enough to make differences, especially in the placement of x^2. In many cases, it was difficult to assess whether the student had solved the problem correctly. This suggested that the problem (and its solutions) might change, if the question included the phrase "estimate as accurately as possible"? However, the results from administering this second modification to another group of pre-calculus students were not significantly different from our earlier results.

We found that several students placed the values correctly, but then offered flawed explanations. This casts some doubt on the validity of identifying all those responses in category 6 as correct. For example, one student correctly located all the points, but described the location of x^2 as "the distance between 1 and x doubled." Another student suggested that "x^2 is right next to the # below it which is x."

Others placed the points incorrectly, although possibly close, but were able to offer correct explanations. One student correctly located 2, 2x, and x^2, but placed 2^x

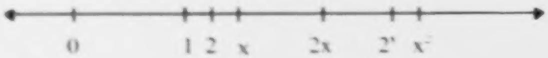
incorrectly. This student explained: "Based upon the location of x, I estimate it to have a value of 1 1/3 therefore I plugged this value in for x in x^2 and 2^x to find their respective solutions and plotted the numbers accordingly." In this case, it would appear that the student understood the unit value of one since she made a reasonable estimate of the value of x in order to calculate and locate x^2. However, her location of 2^x was incorrectly located.

Discussion

There were two approaches that seemed to be used by the students in completing this task. The first strategy was estimate, calculate, and then estimate. This strategy seemed to be the one used by the most successful students most frequently. These students located 2 by either an additive strategy (the same distance from one as one is from zero) or a multiplicative strategy (twice the distance as one is from zero); some simply placed 2 correctly without explanation. The remaining values were then calculated, based on an estimate of the value of x as 1.3, 1.4, or 1.5, and placed approximately correctly. It should be noted that this strategy at a representational level requires the student to move from the graphical to the numeric and then back to the graphical to locate the values.

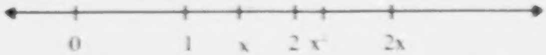
The second set of approaches used involved both additive and multiplicative strategies (and in some cases a combination of the two) for the placement of both 2 and 2x. Some students appeared to use these distance strategies for 2 and 2x and then shifted to an estimate and calculate strategy for x^2 and 2^x. These students used graphical strategies for both 2 and 2x, using either addition or multiplication at an operational level. They then use a strategy that requires shifting from the graphical to the numeric to the graphical, using powers and exponents at an operational level on their calculators. Some students were clearly unable to make this shift in strategy and hence only located 2 and 2x on the number line.

Several significant misconceptions became evident from the qualitative analysis of the students' explanations. We have already noted one such misconception above, namely that x^2 is doubling x. Another striking misconception is the notion that "x" is somehow a label and that it can be assigned an arbitrary value. There is some evidence of this among the very few students who could not correctly locate 2 on the number line. Consider the following student response:



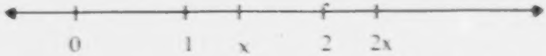
This student locates 2 as indicated and explains that 2 "follows after 1". It then appears that x is taken to be arbitrarily 3: "by plugging in 3 for x, 3 follows 1, 2 is between 1 and 3, 2x = 2(3) = 6, which follows 3 spaces after 3."

Another student places 2 correctly, but the explanation leaves suggests that the role of 0 in determining the unit length may not have been well understood: "I made 2 so 1 and 2 were equidistant to x. I made x = 1.5 because it was half way between 1 &2. I made x^2 1/4 of the way after two because x = 1.5. 1.5*1.5 = 2.25 and I made 2x as if it were 3 2*1.5. The distance from 2 to 2x was equal to the distance from 1 to 2."



Both of these students appear to view x as an arbitrary label that can be assigned a value, rather than as a representation of a value on a number line based on an understanding of unit length.

Another readily apparent misconception is that x measures the distance from 1, that is, that x represents x-1. Many students then interpreted the location of 2x as this same distance added to two, or 2 + (x-1). One student clearly explained: "I found 2x by adding the distance x was from one and making the same distance from 2." Some students continued this interpretation and placed x^2 at about 1/16; others placed x^2 between x and 2.



This student conception has serious implications for how we use the seemingly simple notation of marking "x" on a number line or axis in both calculus and pre-calculus. The data presented here would suggest that a significant number of students would have difficulty in interpreting this notation as well as the frequently used Δx.

Conclusions

As in Pence's study, we found a surprisingly large number of students who were unable to locate 2x on a number line when given the location of x. In contrast to Pence's study, however, we found no correlation between students' success in locating the four values 2, 2x, x^2 and 2^x and their final course grades. An analysis of student's written explanations provided much additional insight into the strategies adopted by students for locating quantities on the number line. It would appear that some students seeing squaring as equivalent to doubling when faced with this particular representation of a variable. Other students could not shift from a graphical representation and graphical actions (either additive or multiplicative) for locating the first two values (2 and 2x) to a coordination of graphical and numeric representations and operations, needed to locate x^2 and 2^x. The most striking finding, however, was the clear indication that many students interpreted x as the distance between x and 1 and then located 2x using an additive operation between 2 and this distance. This finding suggests that a significant number of students are likely to have difficult in interpreting graphical representations and actions that are based on the location of a variable quantity such x and differences such as Δx.

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TEACHING ALGEBRA AND NEGATIVE NUMBERS: TWO CASE STUDIES

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This article presents two case studies where two students from the second grade of secondary school achieve the extension of the numerical domain of natural numbers to that of whole numbers when they solve algebraic problems and equations. However, both students manifest *the predominance of the negative* and *a lack of equilibrium between the semantics and syntax of algebraic language*. These phenomena obstruct the processes of substitution and generalization in more complex situations.

INTRODUCTION

A research project was recently carried out whose central problem was the study of the interrelationships between the following categories: a) processes of acquisition and use of algebraic language; b) methods of solving word problems and linear equations; and, c) status of the negative number in word problems and linear equations (Gallardo, 1995, 1994 and Gallardo & Rojano 1994, 1993). The general methodology of the project dealt with the interaction of these categories on two levels, the historical and the didactic. On the historical level, chapters from ancient texts which provide some evidence of the status of the negative number in the context of problems and equations were studied. Furthermore, the different levels of language were studied as well as the methods and strategies used in problem solving. The historical period studied mainly comprises the XIII and XV centuries, although historical antecedents were also examined (in the Chinese, Greek, Hindu and Arab cultures). On the didactic level, an experimental design with students from the second year of secondary school was formulated. The basic content of algebra teaching in secondary school corresponds to the historical period analyzed. The experimental design included the use of questionnaires and video-recorded clinical interviews.

In the historical context it was concluded that the extension of the numerical domain of natural numbers to that of whole numbers is achieved when there is a syncopated language of expression, an algebraic method of problem solving, fluid operativity and the interpretation of negative numbers as signed numbers, relative numbers and isolated numbers. These conditions are also met in a didactic context. However a more detailed study shows that once the extension of the numerical domain is achieved, the *Predominance of the Negative* and the *Lack of Equilibrium between the Semantics and Syntax of the algebraic problem* persist and make it difficult to let go of the significance of the symbols and make the process of substitution in equations and problems meaningful. These phenomena were defined on the basis of the evidence found in the case studies.

TWO CASE STUDIES

The students selected were the highest achievers of a population of 35. The reason for choosing two students with a high level of performance was to analyze "how far a secondary school student can reach" regarding the conceptualization and operativity of negative numbers.

When they are presented with problems which are very difficult for their grade in school, the students manifest the phenomena mentioned above. We will now describe what happened in one of the case studies.

The phenomenon *predominance of the negative* means that the presence of a negative number in a expression overpowers natural numbers. For example, when a student is asked to say that $(-1)^n$ is equal for $n = 1, 2, 3, \ldots$, he shows the predominance of the negative. He says that $(-1)^n$ will always be equal to minus 1. Decoding $(-1)^n$ as $(-1)(n)$ also emerges. Both erroneous interpretations disappear when it is suggested that the students evaluate the expression for $n = 1$, $n = 2$. In the following exercise, $(-a)^n = ?$, $n = 1, 2, 3, \ldots$, the student wants to decode the a as a positive number or a negative number. The interviewer tries to convince him that a does not have to be substituted by a numerical value. In this more complex situation the student does not accept the permanence of a as an unknown number, he cannot conceive of it and wants to evaluate it immediately.

The student is then presented with,

$((-a_1)(-a_2)\ldots(-a_r)(\ ((+b_1)(+b_2)\ldots(+b_s)] = ?$, $r = 1, 2, 3, 4, \ldots$ $s = 1, 2, 3, 4, \ldots$

In this exercise the sign of the product cannot be found without evaluating the a's. The student is concerned as to whether they are positive or negative. With help from the interviewer he accepts not evaluating the a's numerically and finds for the particular case of $r = 3$ that

$$(-a_1)(-a_2)(-a_3) = -(a_1)(a_2)(a_3)$$

and for $r = 4$ that

$$(-a_1)(-a_2)(-a_3)(-a_4) = +(a_1)(a_2)(a_3)(a_4)$$

With the intervention of the interviewer, the student writes:

$(-a_1)(-a_2)(-a_3)(-a_4) = (-1)^4(a_1)(a_2)(a_3)(a_4)$ that is, he describes the sign of the expression as -1 to the subindex of the last factor. The idea is that he achieves generalization on the basis of the particular cases $r = 3$, $r = 4$. The student believes that if the expression is written with $(-1)^r$ in front of the factors a_1, a_2, a_r, this means that $a_1 = a_2 = a_r$. Again he makes the mistake $(-1)^3 = 3(-1)$. Once it is clarified that a_1 (a_2 (a_r, he does not understand "the point of" writing -1 as $(-1)^r$. Finally, with the help of the interviewer, he writes $(-a_1)(-a_2)(-a_3)\ldots(-a_r) = (-1)^r(-a_1)(-a_2)\ldots(-a_r)$. The student does not see that the minus signs are found in $(-1)^r$ and repeats them again accompanying the *a*'s.

The lack of equilibrium between the semantics and syntax of algebraic language is found in problems with negative solutions. The statement of one of the problems is: *The square of the fifth part of a group of monkeys minus three, three are hiding in a cave and one of them is in the tree. How many monkeys are there?*

In this part of the interview the student has become a good user of algebraic language. The first thing he does is to name the unknown. He formulates the equation correctly. With the help of the interviewer he obtains the solution $x_1 = 5$, $x_2 = 50$. He rejects the first solution, arguing that "you can't have minus 2 monkeys". In replay to the interviewer's question: "What do you mean?", he says: "Yes, because 5 divided by 5 is one, minus 3, minus 2. Thus, there can't be minus 2 monkeys in a group" (the student substitutes the value 5 in the equation $\left(\frac{x}{5} - 3\right)^2 + 1 = x$).

The other case study shows in the following situations the *Predominance of the Negative*:

1. The student affirms: "This sign outside [- (-3) = ?(indicates to us that it is a negative expression. Consequently, the number remains negative although minus times minus is plus".
2. He decodes 2 - (-3) as 2 + (-3) or 2 - (+3). The student appeals to the multiplicative rule of signs but the minus sign persists.
3. He interprets 1 - 2 as 1 - (-2), giving the following explanation: "... because when substracting and you are in the left part of the number line, the number becomes negative".
4. In $(-1)^n = ?$, $n = 1,2,3, \ldots$, the student affirms: "$(-1)^r$ is equal to minus 1, because when you multiply all numbers, they are going to result equal to minus 1".
5. In $(-a)^r = ?$, $n = 1,2,3, \ldots$, the student writes $(-a)^1 = -a$; $(-a)^2 = -a^2$; $(-a)^3 = -a^3$. In $[(-a_1)(-a_2) \ldots (-a_r)([(+b_1)(+b_2) \ldots (+b_s)(= ?$ $r = 1,2,3, \ldots$; $s = 1,2,3, \ldots$, he confuses subindexes with exponentials. Interviewer explains the mistake and draws the student to notice that the point is to determine the sign of the result. Then, the student writes correctly the expressions:

   $(-a_1)(-a_2)(-a_3)(-a_4) = +(a_1)(a_2)(a_3)(a_4)$

   $(-a_1)(-a_2)(-a_3)(-a_4)(-a_5) = -(a_1)(a_2)(a_3)(a_4)(a_5)$

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The interview continues as follows:

Interviewer: Without solving it, could you tell me which is the sign of the following expression?

$(-a_1)(-a_2)(-a_3)(-a_4)(-a_5)(-a_6)$?

Student: Positive.

Interviewer: And the next?

Student: Negative.

Interviewer: Then ...

Student: r is negative, thus the result is negative.

To consider r as negative is a symptom of the *Predominance of the Negative*. This phenomenon arises when posing the following question:

If m and n are whole numbers, is (-m) (n) a positive whole number? The student afirms, "minus m times n is always negative".

With regard to The Monkey Problem mentioned before, the student appeals spontaneously to algebraic language. After proving the validity of the solutions by substituing in the equations, the students afirms: "There are two solutions, and the one which seems more coherent and logic to me is: 50 monkeys because herds are big". The context influences and decides the correct solution to the problem.

CONCLUSIONS

From the two case studies we can conclûde that students with excellent achievement in the tasks corresponding to their grade in school have problems in more complex situations involving negative numbers. These are:

- *The predominance of the negative*. The presence of a negative number in general statements overpowers natural numbers. The student says that the result will always be negative.
- *The need to evaluate numerically all symbols impedes the process of substitution in general expressions*. In $(-a)^n = ?$, $n = 1, 2, 3, \ldots$, the student does not decode the a as an unknown number. He does not conceive of a process of partial substitution where some literals are evaluated and others are not.
- *Obstruction in semantic interaction - syntax of elementary algebra*. Of relevance is the spontaneous use of algebraic language in more complex situations. The student attributes total validity to the method of substitution and the process of verification. However, in the Monkey problem, the context prevents accepting one of the two positive solutions. This happens during the process of verification in the first case study where the subject cannot abandon the meaning of the symbol (monkeys) and gives sense to the process.

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A TWO-YEAR COLLABORATIVE ACTION RESEARCH STUDY ON THE EFFECTS OF A "HANDS-ON" APPROACH TO LEARNING ALGEBRA

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This paper describes a two-year collaborative action research project in which the focus was to investigate the effects of the use of "Hands-On Equations"[1] mathematics manipulatives in an algebra class on students' confidence, interest in, and ability to solve and retain understanding of algebraic equations. The first phase centered on documenting and comparing approximately 120 urban students' reactions to and accomplishments during both manipulative and "textbook" approaches to learning algebra in their eighth-grade classrooms. The second phase of the study was a follow-up on these same students regarding their retention of the manipulative "algebra learning strategies" during their ninth-grade mathematics experiences. Data collection methods included surveys, student reflections, work samples and test scores, and interviews. In brief, findings from phase one indicate students' confidence, interest, and ability in solving algebraic equations were very high when working with manipulatives. In addition, results of a mandated corporation-wide standardized algebra test far exceeded the corporation's expectations.

Collaborative Action Research

Research suggests that collaborative action research, in which classroom teachers and university researcher work together in the investigation of classroom phenomenon, provides a medium for teachers to systematically look at the problems or questions they face in their classrooms in an effort to find practical solutions (Cardelle-Elawar, 1993; Miller & Pine). A number of goals of collaborative action research have been identified, including stimulating classroom reform, improving teaching and learning, providing opportunities for teacher enhancement, and generating theory and knowledge (Raymond, 1996). There is much debate about whether or not action research should be valued beyond its teacher enhancement and reform opportunities. Noffke (1994) contends that one of the contemporary challenges of action research is to address the question of whether findings from action research can truly contribute to the body of research on education and educational reform or whether action research must be considered a singular form of research with methodologies unique to the field.

Reported herein is a description of a two-year collaborative action research project in which an eighth-grade teacher, Marylin, questioned the worthiness of her efforts to teach algebraic concepts via manipulatives. Although the initial motivation for the study was to critique the quality of mathematics teaching and learning in Marylin's algebra classroom, the project reveals research findings worthy of consideration by the theoretical community as a welcome addition to the modest body of literature on the teaching and learning of algebra (see Kieran, 1992). What follows is a brief description of the project structure and results from the first phase of the collaborative mathematics study.

Motivation for the Study: Marylin's Dilemma

Marylin had a dilemma. Because her middle school had elected to implement an "algebra for all" program, she suddenly found herself, after teaching middle school mathematics for 22 years, faced with teaching algebra to a wide range of students, many of whom would not have normally been encouraged to pursue algebra. She was not confident that she would be able to efficiently teach algebra to all of the students through a traditional textbook approach. Fortunately, she was afforded the opportunity to attend a workshop on teaching algebra through a manipulative-driven algebra program.

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[1] "Hands-On Equations" was developed by Dr. Henry Borenson.

As Marylin began to teach the innovative program during the tenth week of school, she started to worry about whether or not the program was effective. Specifically, she was concerned that some students would become dependent on the manipulatives and might not fare well when faced with traditional teaching methods in future high school mathematics courses. She was also concerned about the depth to which students would learn algebra and how they would perform on the end of the year standardized algebra test required to be taken by all algebra students, particularly since her colleagues at the middle school and high school taught strictly via algebra textbooks and were skeptical of her hands-on teaching practices.

It was at this point that she met with Anne, a university researcher, to discuss ways to investigate the following questions: (a) How does the use of these mathematics manipulatives in an algebra class effect students' confidence and interest in solving algebraic problems? (b) How does the use of these mathematics manipulatives in an algebra class affect students' ability to correctly solve algebraic equations and (c) Will the students' retention of algebraic skills learned via manipulatives last beyond the eighth-grade experience?

Methodological Procedures

Phase One

This study had two phases, the first taking place during the 1994-95 school year. For the first nine weeks of the school year, Marylin taught in a non-manipulative style using the adopted textbook. Following this nine-week period, she implemented the 26-lesson manipulative program. In short, the materials in this program introduce students to a manipulative approach to solving algebraic equations, and guide them through an intermediate pictorial approach, culminating in engaging students in activities that relate the manipulative to the more formal "high school" algebra. The reader needs to be aware that students were allowed, and encouraged, to use manipulatives during quizzes and tests given during the manipulative program. The tests and quizzes were designed in a format that paralleled the manipulative instruction.

The subjects of the study include five classes of eighth-grade students, approximately 120 students, at a lower class, inner city middle school in Indiana. Data collection methods include an end-of-year survey (see Appendix), weekly student reflections, teacher observations and teacher reflections, student work samples and test scores, and a whole-class interview (conducted solely by the university partner). These interviews focused on students' confidence and interest in learning algebra when working with mathematics manipulatives versus working with a textbook.

Data about students' ability to solve algebraic equations were initially gathered through student work samples and student tests scores during both the "manipulative phase" and the "book" phase. Some students were also videotaped while demonstrating algebraic solutions during class time. Transcriptions of these videotapes serve as a verifying source of data on students' abilities in algebra. Additional data on students' ability to solve algebraic problems was gathered from a mandatory standardized algebra test given to all eighth-grade students in the middle school at the end of the school year.

Phase Two

The second phase of the study took place over the course of the 1995-96 school year, during which time we continued our investigation of these same students who have moved on to high school. We were interested in ascertaining the "durability" of the results of the manipulative experiences in phase one. In March 1996, surveys were mailed to approximately 90 students who could be located. Only 19 completed surveys were returned. Of those who completed the survey, eight students indicated that they would be willing to participate in a one-on-one interview during the summer of 1996 to talk about the past two years.

Findings From Phase One of The Study

Test and Quiz Grades

We first compared overall class grade averages from the textbook phase to those earned during the manipulative phase. Table 1 shows these initial results. In each case, overall class averages were higher during the manipulative phase than the textbook page.

Table 1

A comparison of class averages during periods of textbook instruction and manipulative instruction.

	Class Average	
Class Period	Textbook	Manipulatives
1st	65%	82.38%
2nd	70.47 %	81.28%
3rd	75.07%	85.29%
6th	81.4%	87.82%
7th	72.16%	82.1%

In general, individual students scores were higher during the manipulative phase than during the textbook phase. Some of the differences were quite significant. For example, 23% of the students went from below "C" scores to scores of 70% or higher, and 42% of the students earned an average of "A" work on their algebraic work with manipulatives whereas only 14% earned "A's" during the book phase. On the other hand, 12.5% of students did not have higher scores during the manipulative phase. Of these students, 33% had below "C" scores during both phases and 60% had a percentage difference between manipulative and book phase scores of less than 5%.

It is difficult to conclude what these numbers tell us. Certainly for Marylin, the results are meaningful to her practice in that the percentages provided some indication that students could solve algebraic problems well with the aid of manipulatives. Thus, it was clear that many students were better able to demonstrate their abilities through the manipulatives and were able to show understanding of algebraic concepts via the manipulatives. On the other hand, these percentages also cause some concern that perhaps the students may have "needed" the manipulatives to show what they know. Also, since work with the book came before and after the work with manipulatives, it is unclear to what extent the manipulative experience influenced later textbook performance. Thus we were compelled to break down student scores further (see Table 2).

Table 2
A comparison of class averages during periods of manipulative work and textbook work before and after manipulative instruction

Class Period	Class Average Textbook Before	 Manipulatives	 Textbook After
1st	78.36%	82.38%	57.13%
	77.82%		68.93%
2nd			
	76.07%		75.4%
3rd			
	87.24%		77.65%
6th			
	74.96%		70.33%

Clearly, in every case the class average during textbook instruction decreased after the manipulative instruction period. As before, individual achievement varied. For example, 77% of the students showed a decrease in individual average on textbook work after the manipulative phase. Individual results varied from class to class. All of the students in the first period class showed a decrease, while only 48% of the students in the seventh period decreased. Of all students who showed a decrease, 47% of those students' scores decreased by more than 10% and 21% earned a score of "D" or less in both textbook periods.

Marylin was quite disturbed by these results. A primary concern was that the manipulatives had weakened the students' abilities to work algebraic problems without manipulatives. She also considered the possibility that the students had not retained the learning they had achieved during the manipulative phase. Another possibility was that students may not have been making the connections between the concrete learning and the more abstract learning. Also, it could be that the material in the later part of the textbook was more difficult than earlier material, resulting in lower scores. And yet another likely possibility was that students did not enjoy the work from the textbook as much work with manipulatives and thus, did not put as much effort into their work. Any or all of these conclusions could be valid. However, the real dilemma for Marylin became how did the students explain the reasons for the decrease and how could she change her practice to deal with these unanticipated findings? These questions formed the foundation of questions to ask the students during phase two.

Standardized Test Performance

At the same time Marylin was agonizing over the surprising class averages, an additional piece of data provided a positive twist. On the standardized algebra test given to all eighth-grade students at the middle school, Marylin's students performed satisfactorily, far exceeding the expectations of the administration and colleagues. Approximately 80 % of Marylin's students correctly responded to approximately 60% of the test questions. Because test questions were worded and had to be solved in a traditional algebraic fashion, Marylin believed she had successfully helped students bridge the gap between concrete and the more abstract algebra, even though the comparative textbook data left some measure of doubt as to the degree to which students fully made the connections.

Phase One Survey Responses

On the first-year survey, students were asked a variety of questions regarding their interest in learning algebra. Only 68 of the 120 students returned completed surveys. Responses to the survey question, "How did you feel when you learned that all eighth-grade students would have to take algebra?", students provided answers such as:

> I thought it was going to be really hard .... I didn't want to take algebra. I'd rather take basic math.... I felt a little scared....I felt that doing algebra in the eighth grade would be fun....I felt like "oh no" I'm going to fail this class....I had never heard of it, but glad I did....I felt kind of intimidated by it because I thought it was a high school course....Mad, because I [stink] at math.... I thought I was going to be grounded every time a report card came out.

However, when asked how they felt about algebra after finishing the manipulative lessons, they expressed:

> I felt that it was a neat experience and that it wasn't so hard after all....Relieved, algebra was a breeze ...I really liked it because it was the most fun....I kind of liked it because it got easier as the lessons went on....I felt that I had learned more by the manipulative....Very comfortable about algebra....Good, the manipulatives were fun and very helpful....I felt it was easier because some of the things we did in manipulatives we could transfer into our regular algebra....I found it easier than the book....I didn't want to leave the manipulatives, they were fun.... I proved I could when I thought I couldn't.

When asked which approach to learning algebra they liked better, using the textbook or working with manipulatives, 91% of the students preferred the manipulatives. Explanations they provided included:

> There was no homework...using your hands...it was easier and funner...I always scored high... they were easier to understand...it wasn't boring...it was new... we didn't read much...we did really hard problems...everyone got involved...the textbook was hard and skipped around a lot... working with things other than texts make you alive and ready to work... I learned quicker with the manipulatives...you could actually see what you were doing...

Those who preferred the textbook explained:

> manipulatives were too messy...with the textbook you didn't have to worry about putting things up...it was more organized

Fifty-seven percent of the students expressed that they learned more algebra when working with the manipulatives. They suggest:

> We spent more time on it and it was funner...made me want to learn more...we could learn faster...because you see how you get the answer... when you actually touch the problem it's easier...I did more problems with the manipulatives...

Those who said they learned more when using the textbook reasoned:

> The teacher explained more to us...the book will always have more learning ......it was more in depth... because we were in the book longer...book explained more...doesn't take as long...more detailed...had more difficult problems...the textbook you could read the pages to understand...I remember it more...I learned more because we took notes and listened to lectures...

Conclusion

Thus far, the data suggest that most of the eighth-grade students performed better academically and expressed more positive attitudes about algebra when working with manipulatives as opposed to the text. However, many unanswered questions remain at the end of phase one. We are intrigued by the initial findings and look forward to learning more about the long-term effects of this innovative teaching approach as phase two unfolds.

Collaborative action research provides an additional layer of professional enhancement "results" beyond the findings related to the focus of the inquiry (Raymond, 1996). Engagement in the inquiry process impacted Marylin's practice by causing a great deal of informed reflection to take place. Marylin continues to question her mathematics teaching practice and actively seeks ways to investigate and document the successes and limitations of her teaching. Thus, not only did this action research study provide a window through which to critique the results of alternative methods of teaching algebra, but it also encouraged reflective mathematics teaching.

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Appendix

Examples of Survey Questions- Phase One

How did you feel last August when you learned that all eighth graders would take algebra?
How do you now feel about algebra at the end of the school year?
Rate your knowledge of algebra? (Check one) Low __ Medium __ High __ Explain your rating:
Rate your confidence in doing algebra? (Check one) Low __ Medium __ High __ Explain your rating:
Using Dr. Borenson's "Hands-On" Equations method, solve the following equation with the "pictorial" method and explain how you did it: $2x + x - x + 1 = x + 9$
Which did you like better?: (Check one) Using the textbook __ Working with manipulatives __ Explain:
When did you learn more algebra?: (Check one) With the textbook __ With manipulatives __ Explain:
What would you change about the eighth-grade algebra program?

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INNER CITY MIDDLE SCHOOL STUDENTS' UNDERSTANDING OF SPEED[1]

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Inner city middle school students engaged in a year-long curriculum revolving around participation in community-based mathematics projects, which were designed to strengthen their mathematical problem solving/modeling abilities, especially with regard to ratio, rate and proportionality. Students were given a pre-test and post-test to assess the growth in their understanding of the concept of speed. Responses to the problems were coded according to strategies used. Results indicate that although the students improved their understanding of constant speed, their understanding was not general enough to successfully analyze a novel situation in which speed was not constant.

There is general agreement among mathematics educators that the understanding of rate and ratio as well as the ability to reason proportionally is fundamental to algebraic thinking (Herscovics, 1989). The ability to think algebraically is widely regarded as an important outcome for all students (NCTM, 1989; Pelavan & Kane, 1990; Thompson, 1994).

The difficulties that middle school students have with the concepts of ratio, rate, and proportion are well documented (Harel and Confrey, 1994; Herscovics, 1989). Recent research suggests alternate ways of teaching and assessing these concepts (e.g. Cramer, Post, & Currier, 1993). One direction that holds promise rests upon a theory of situated cognition (Lave, 1988), and upon the ubiquity of rate, ratio and proportion in everyday life. The research on situated cognition formed the basis for the instructional approach used in this study. Inner city middle school students engaged in a year-long curriculum centered around participation in community-based mathematics projects designed to strengthen the students' mathematical problem solving/modeling abilities, especially with regard to ratio, rate and proportionality.

Speed is both familiar and intriguing for middle school students. As a result many middle school mathematics teachers rely on speed as a pedagogical tool for teaching ideas of rate, ratio, and proportion. In this study we used two-column (function) tables of time and distance as a basis to assess middle school students' growth in understanding of the concept of speed. Student assessment was done before and after engagement in this year-long mathematics curriculum, which revolved around community-based mathematics projects.

METHOD

Participants. The participants were 220 sixth-, seventh-, and eighth-grade students and their two mathematics teachers from a public school in a large urban school district. The students were ethnically and culturally diverse, with 84% of the students qualifying for free or reduced lunch. Fifty-seven percent (57%) were African American, 20% were Hispanic, 15% were Caucasian, and 8% were "other." The two mathematics teachers who participated in the study did so on a volunteer basis. One teacher was highly experienced, having taught middle school mathematics for more than 15 years. The other teacher was relatively inexperienced, possessing an undergraduate major in music and education, and having previously taught mathematics for one year. The two teachers relied on each other for support, and teamed together for planning purposes.

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Instruction. Throughout the school year, two university professors met with the two project teachers regularly to develop community-based projects which were designed to strengthen the students' mathematical problem solving/modeling abilities, especially with regard to ratio, rate and proportionality. The recognition and representation of patterns and relationships was a recurring theme. As the year progressed the projects themselves became more open-ended, more complex, and longer in duration. As a culminating project, each student spent approximately twelve weeks at the end of the year engaged in one of four community-based projects.

Pre-tests and Post-tests. In October, 1995, a pre-test of four tasks designed to assess the students' understanding of constant speed as a function relating distance and time was administered. In May, 1996, a post-test of four tasks similar to the pre-test tasks was administered, together with four new tasks designed to assess the students' understanding of non-constant speed. The post-test is given in Figure 1.

Data Coding and Analysis. Responses to questions were coded according to a preliminary classification scheme motivated by Piaget (1970) and later modified to better reflect the actual strategies used by the students on the pre-test. General category descriptions are displayed in Table 1. The strategies in category 2 can be viewed as intermediate level strategies, which students used successfully in solving some types of speed problems, and not others. Data were recorded for the 150 students who took both the pre- and post-tests, and frequencies were tabulated by category (see Table 2).

RESULTS

The problems on the pre-test and on the post-test part 1 were the same except for different numerical data. The data show gains in the level of sophistication of the strategies used to answer questions about speed. This can be seen from two perspectives: First, overall in the post-test students seemed to better understand the constant speed situation, as presented in the form of data tables. Question 1a on the tests was a very simple "warm-up" question designed to give the students confidence in themselves. In fact, 104 of the students were able to correctly answer the question, even though their explanations indicated a low-level (category 1) strategy. Because of this, we did not conduct a chi-square test for differences on the results of 1a. However, we did conduct chi-square tests on the results of 1b, 1c, and 1d. Significant differences between

Figure 1 Post-test

1. José and Tasha have bicycles with odometers that measure distance traveled.

On a bike trip, José began by resetting the odometer to 0, and on the trip he recorded the distances every two minutes. The table below shows the data José collected.

Tasha began by resetting the odometer to 0 and recorded her distance every 5 minutes. The table below shows the data Tasha collected.

José's Table

TIME (minutes)	DISTANCE (meters)
2	300
4	600
6	900
8	?
10	1500

Tasha's Table

TIME (minutes)	DISTANCE (meters)
5	600
10	1200
15	1800
20	2400
25	3000

a. José forgot to record the distance he traveled in 8 minutes. Based on José's table, find the distance José traveled in 8 minutes. Explain how you found your answer.

b. José noticed that Tasha traveled 600 meters in 5 minutes. José wanted to know how far he traveled in 5 minutes. Find out how far José traveled in 5 minutes. Explain how you found your answer.

c. José and Tasha wanted to know who traveled faster. Based on the tables, figure out which one of them traveled faster. Explain how you found your answer.

d. Based on Tasha's trip, find the distance Tasha traveled in 12 minutes. Explain how you found your answer.

2. James and Maria are runners on the track team, training for distance running. One day, James ran a 10,000-foot race in 19 minutes, and Maria ran a 7,000-foot race in 14 minutes. The coach recorded data about each race in the tables below.

James' Race

TIME (minutes)	DISTANCE (feet)
1	400
3	1,200
5	2,000
7	2,800
9	4,000
11	5,200
13	6,400
15	7,600
17	8,800
19	10,000

Maria's Race

TIME (minutes)	DISTANCE (feet)
2	1,000
5	2,500
8	4,000
11	5,500
14	7,000

a. Did James run at the same speed the whole time? Explain how you found your answer.

b. Did Maria run at the same speed the whole time? Explain how you found your answer.

c. Who was running faster as they crossed their finish lines? Explain how you found your answer.

d. Overall, did James or Maria run the faster race? Explain how you found your answer.

Table 1 General Categories of Responses by Strategy Type

Category	Description
0	No response, or no sensible strategy used
1	Strategy makes use of a pattern in the distance data only; no coordination of distance data and time data is explicit in the strategy
2	Strategy reflects some coordination of the data in the tables, but this coordination may be qualitative only, emphasizing the numerical pattern in the data, or not explicitly using the rate nature of the coordination
3	Strategy makes explicit use of either (1) a composite unit of distance and time in building up the table of data, or using the correspondence of a particular distance increment with a particular time increment, or (2) a unit rate, either in building up a table, or used with multiplication

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Table 2 Frequency Of Responses

Cat	Pre-test				Post-test: Part 1				Post-test: Part 2			
	1a	1b	1c	1d	1a	1b	1c	1d	2a	2b	2c	2d
0	11	44	49	64	15	40	36	35	93	94	52	69
1	104	61	45	61	92	47	34	73	23	26	46	48
2	10	15	46	11	11	20	69	13	21	9	42	21
3	25	30	10	14	32	43	11	29	13	21	10	12

performances on the pre-test and post-test were found for each of questions 1c and 1d, but not 1b (1b: $c^2=5.035$, $p=0.169$; 1c: $c^2=8.167$, $p<.05$; 1d: $c^2=14.969$, $p<.01$). However, on question 1b more students on the post-test (29%) than on the pre-test (20%) could use a level 3 strategy to solve the problem. Similarly, on question 1c more students could correctly compare the speeds of the two bikers (using a level 2 strategy) on the post-test (46%) than on the pre-test (31%). On question 1d more students on the post-test (19%) than on the pre-test (9%) could use a level 3 strategy to make predictions about distance at a particular time.

Second, we compared the strategies each student used on the pre-test with the strategies they used on the post-test. To do this, we compared first the strategy categories used on question 1d. If no change occurred from pre-test to post-test, then we compared the strategy categories used on question 1c. Finally, if no change took place on question 1c, the strategy categories used on question 1b were compared. Using this scheme, we found 86 of 150 students (57%) used a higher category strategy in the post-test than in the pre-test, 43 (29%) declined, and 21 (14%) exhibited no change.

DISCUSSION

Category 1. We found that most students could make observations about the distance patterns (strategy category 1) in a biker's trip (question 1a) and correctly answer the question, but may or may not have understood explicitly that time played a role in the speed situation. Category 1 strategy also was often successful in using a "half-way" interpolation to determine how far the biker had traveled at 5 minutes, given data about 4 minutes and 6 minutes. The use of interpolation, whether successfully or unsuccessfully (e.g. using an additive method), implies that the subject was aware that the time values somehow corresponded to the distance values, but was not using this fact explicitly. It is also not clear whether the student understood the speed situation, i.e. he or she may just have used number patterns to answer questions.

If we contrast this question with questions 2a and 2b (in post-test part 2), we see that many fewer students were able to make judgments about speed in this unfamiliar situation in which the speed does not remain constant. (These types of problems were not explicitly studied during the course of the year). Only 46 students were able to note in question 2a, that James changed speed midway through his run; 56 students noted in question 2b that the Maria did not change speed during her run. Thus, although in 1b of the post-test 40 students (27%) were unable to give any sort of reasonable answer (category 0), for post-test questions 2a and 2b, 93 (62%) and 94 (64%) students respectively were unable to answer reasonably.

Category 2. This strategy on speed comparison problems involved comparing distances at common times in the data tables or times at common distances. Thus, this strategy is adequate to make speed comparison judgments when the speed is constant, but not when speed is changing, as in post-test questions 2c and 2d. Use of this strategy also seems to reflect (for some) a conceptualization of speed as a way to make relative comparisons between proximate simultaneously occurring events, rather than as an absolute measure of motion which can be used to compare events separated in space and time. Many students' responses indicate they conceptualized the speed comparison question 1c as a question about who won a race (even though the situation does not make reference to a race).

We see this conceptualization again in attempts to compare speeds in questions 2c and 2d in post-test part 2. Question 2c asks which runner was going faster "across the finish line" (i.e. a question about speed over a short interval of time), and question 2d asks which runner had the fastest overall speed (average speed). Without using a category 3 strategy, a successful response on these questions is not to be obtained. Students who attempted to compare distances at common times or times at common distances used data near the beginning of the races, irrelevant to finish line speed, and neglected the fact that the James sped up part way through his run (even though they may have made this observation earlier). A few students noted that the runners' speeds could not be compared, because they "didn't run the same race". Some were able to find a unit rate (500 feet per minute) for Maria (who maintained a constant rate), but didn't know what rate to use for James. Calculating average speed for the run in the case of changing speed was not an obvious choice for these students.

Category 3. Even though overall, many students showed gains in general sophistication in understanding the constant speed situation, few applied successful thinking to the changing speed situation. Because questions 2c and 2d involve changing speed situations, they clearly require strategies beyond comparing distances at common times or times at common distances. There were a large number of students who failed to use an appropriate strategy on either of post-test questions 2c or 2d despite success in comparing speeds in the constant rate situation. We believe this result may indicate that these students do not yet conceptualize speed as a quantification of motion, but rather as a way of comparing two motions relative to one another. For them, speed seems to be a way of making qualitative, comparative judgments ("José wins, or goes faster") rather than quantitative, absolute measurements.

Of the 150 students, only 15 (10%) correctly answered questions 2c or 2d using a category 3 strategy. The use of a category 3 strategy on both questions indicated that the students successfully distinguished between overall average speed and speed over a short interval near the end of the run. Less than half of these 15 students were able to successfully distinguish these two types of speed. Seven students got both questions correct; 3 got question 2c correct but not 2d, and 5 got 2d but not 2c correct. These results indicate that the similarities and differences between the two types of speed are not easy to reconcile even for students who exhibit an understanding of speed as a quantitative, absolute measurement of motion. Perhaps this is due to the fact that the students are seldom asked to distinguish between them. That is, most rate situations encountered in the normal curriculum (as well as in the project-based curriculum of this study) involve constant rates. Since, in the constant speed situation, both overall average speed and speed over a shorter interval always yields the same numeric result, students usually do not come to regard them as different.

Conclusion. The results of this study indicate that speed is a complex concept admitting to many levels of understanding. Strategies which students use successfully when dealing with simple speed situations are not necessarily successful when applied to more complex situations. Students who are involved in community-based projects designed to strengthen their mathematics problem solving/modeling abilities, especially with regard to ratio, rate, and proportionality, may improve their understanding of the concept of speed. However, unless specific attention is paid to increasing the complexity of rate situations within the community-based projects, students may not reach an understanding of speed that is general enough to be successfully applied across a wide variety of situations.

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MAKING SENSE: CHILDREN INTERPRET EQUATIONS AND THE EQUAL SIGN

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Second grade students rejected equations such as $3 = 3$ or $6 = 4 + 2$, saying, "It doesn't make sense." Similar responses, and procedural interpretations of the equal sign, have been reported previously (Behr, Erlanger & Nichols, 1980; Herscovics & Linchevski, 1994). This study investigated how students made sense of equations and the equal sign.

Existing knowledge is the basis for making sense of symbols. Children's first mental representations are of sequential scripts for events. The meaning of a concept embedded in such a script is first limited to the position it holds in the script; later, children abstract the concept (Nelson, 1983). Analysis of whole class and interview transcripts from the first phase of this study suggests that these students made sense of equations by comparing them to a common sequential event they called a "math problem." In the event's written form [$a \pm b =$ __] the equal sign was just a mark at the end of the problem. Students demonstrated a rich conception of numerical equality. The equal sign did not represent equality to them as they tried to make sense of equations.

In the second phase of the study, students engaged in manipulative activities involving novel equations. They used the equal sign to compare quantities. Students' constructions were documented in whole class, small group, and interview contexts. Children subsumed their rigidly sequential "math problem" into the more general "math sentence" and incorporated the conceptual, equivalence-based meaning for the equal sign into the sentence. Some students made sense by calling on their increased script-based knowledge and others made use of the conceptual meaning of the equal sign. All made sense of previously rejected equations and used the equal sign as a symbol of equality.

A child's ability to read and operate on written symbols cannot be assumed to imply that child's conceptual use of each of those symbols. A child cannot make sense of a written mark in a new context if the sole meaning of that mark stems from its usual position in a familiar script. Once a conceptual meaning has been abstracted for the mark separate from its procedural position in the script, the mark becomes a conceptual symbol. The concept is then available to the child for making sense of new equations.

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A CONSTRUCTIVIST APPROACH TO TEACHING SLOPE

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The purpose of this study was to investigate a constructivist approach to teaching slope concepts. This approach builds on students' intuitions about steepness, setting as its goal that students articulate and formalize their intuitions.

Subjects consisted of twelve students, in the 8th grade, from a low SES public school in a mid-sized southern U.S. city. Four one hour sessions were used for the participants to develop the slope formula through grappling with their intuitive understanding of steepness. Session 1 involved open-ended discussions of steepness in which participants were presented with pairs of lines and asked to identify which line was steeper, and to provide reasons for their opinions. There followed a series of exercises in which pairs of participants, seated back-to-back, were required to communicate steepness of a given line using a single number, along with instructions for interpreting that number, to enable their partner to construct a line of exactly the same steepness. Their only supplies were a pencil, paper, ruler, and protractor. In Session 2, the protractor was removed in an attempt to have participants refer to steepness in some manner other than angle measure. After participants were given a sufficient amount of time to grapple with their dilemma, the single number restriction was removed, allowing opportunity for participants to use an unlimited amount of numbers to describe the slope - for instance by giving the coordinates of two points. As participants' thoughts progressed the single number restriction was slowly reinstated. Session 3 involved an exercise intended to raise the issue of relevance or irrelevance of the two reference points chosen on the line when computing the slope number. Session 4 attempted to evaluate participants' understandings of the slope concepts that they had developed.

Most participants displayed a great amount of enthusiasm and satisfaction in their learning of the slope concepts. Unfortunately, teacher-researcher errors during the study allowed a claim of only partial success as a constructivist approach to teaching slope concepts.

FROM WORDS TO SYMBOLS: METHODS OF FIRST-YEAR ALGEBRA STUDENTS

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Purpose and Significance: This study describes the ways that first-year algebra students translate words into algebraic language and their understandings of this process. Translation of natural language into algebraic language is a complex process, and the ways that students accomplish this significant conversion bear scrutiny.

Conceptual Framework: Natural language has a syntax, or grammatical structure, and a semantics, or meaning. A third component of language is its register, or meanings appropriate to a particular function of language, together with the structures that express those meanings (Halliday, 1975). Words, phrases and modes of argument are elements of the language register (Pimm,1987). Fluency in any language requires the acquisition of the appropriate registers and knowledge of what is socially appropriate in linguistic situations. Translation of a problem situation into algebraic language is an obstacle for many students (Clement, Lochhead & Monk, 1981; Greeno, 1982; Kieran, 1989).

Procedures and Design: The indicated methodology for this study was ethnographic Lincoln & Guba, 1985. The theoretical perspective is the Developmental Research Sequence described by Spradley (1980). The population consisted of the students and teacher of one section of Algebra I in the fall of 1995. The school selected was a mid-sized suburban high school. The class selected for study is demographically similar to the school at large.

Analysis of Data: The data from observations and interviews were separated into single units of meaning and grouped (Lincoln & Guba, 1985). Themes emerged as groups were compared and contrasted. Algebraic expressions was selected as the domain of interest. Data analysis was continuous, as was member checking. Interviews continued until redundancy was reached.

Results: Students in this algebra class use key words, grammatical cues, and expressing the relationships between quantities to translate natural language into algebra. These means of translating words to symbols are used singly or in combination by students.

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UNDERSTANDING STUDENTS' REIFICATION OF VARIABLE

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Reification, creating a static cognitive structure from its operational origins, is seen by many theorists as the pinnacle achievement of mathematical understanding (Kaput, 1989; Sfard, 1992). For Sfard (1992) reification also characterizes the process that mathematicians experienced in the historical development of mathematics. The purpose of this study was to refine the methodology through which reification has been studied and to observe reification as it develops.

Fourteen high school students in Algebra II at a university lab school or in the gifted program in a southern, mid-sized city were individually interviewed as they worked through a grid of pairs of equations that varied in their level of computational and structural complexity. Computational complexity refers to the number of computations needed to determine the value of the variable in the first equation. For instance, the pair of equations $3x + 7 = 25$ and $3(x - 9)^2 + 7 = 25$ has computational complexity 2 because two computations, subtraction and division, must be performed to determine the value of the variable, 6. The second equation is derived from the first by substituting an expression in place of the variable in the first equation. Structural complexity refers to the number of operations in the substituted expression. Thus the above pair has structural complexity 2, because the variable, x has been transformed to $(x - 9)^2$. Students' use of the structural relations to solve the second equation by means of the first solution was taken to indicate reification. By manipulating the computational and structural complexity of equation pairs, students could be induced to reify: Increasing computational complexity increases the savings to be realized through reification; decreasing the structural complexity facilitates the observation of structural relatedness.

Some participants persisted in seeing the two equations as unrelated despite the inducements of computational and structural complexity. They seemed to be stuck in a procedural understanding of algebra, with no ability to identify structural regularities. Others were reifiers from the start, utilizing the first equation to its maximum potential, despite surface dissimilarities. The majority of students could be pushed to utilize the first equation in solving the second; though this may have just been step skipping for some students, rather than actual reification of the structural properties of the equations.

WHAT ALGEBRAIC RELATIONSHIPS CAN FIRST-GRADERS DISCOVER? BUILDING MATHEMATICS FROM NATURAL LANGUAGE, RELATIONAL FORM, AND LOGICAL INTUITION

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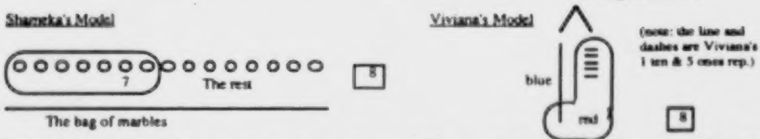
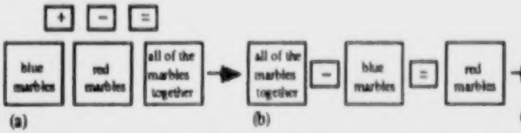
From an inner-city first-grade class that had some experience solving word problems, two girls were selected to work together to explore a next possible step for the rest of the class: using one's own invented model of a specific problem situation to discover all the true statements one could make about that type of situation. The first problem situation they modelled is presented below.

There is a bag of 15 marbles. 7 are red. The rest are blue. How many are blue?



Both girls' models label and mark out the set/subset relationships. That enabled them to point out, discuss and define the components and their interrelationships, then embed the language they found most meaningful on cards (see a). When asked if it would be possible to find what true sentences they could make with them, they intially argued, "But we don't have enough information for this, cause we don't know how many red ones" (Viviana) "Or how many altogether." (Shameka). But when asked, "Even if you don't know how many altogether, could you still make a true sentence?" they quickly discovered and tabulated all possible true sentences and explored deliberately making untrue ones (see c).



(c)

True	Not True
red m + blue m = all m	blue m - red m = all m
all m - blue m = red m	all m + red m = blue m
all m - red m = blue m	blue m - all m = red m

They were also able to explain why statements were true or not true: for blue - all = red, not true, Shameka explains, "Cause the blue marbles are in all of the marbles"; for all - blue = red, true, Viviana explains, "All of the marbles take away the blue marbles leaves only the red marbles". They were able to explore natural language meanings in tandem with relational diagrams, construct logico-mathematical propositions, and explain their truth valuations of them, not depending on numeric exemplars and operations alone. The full paper will report on Vivianna's and Shameka's further explorations of a range of additive problem situations (including Compare and Change situations).

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ASSESSING STRATEGIES AND REASONING IN ALGEBRA

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This study examines how to measure different strategies in solving word problems in algebra. The QUASAR general scoring rubric (Lane, et al., 1994) is evaluated in terms of distinguishing levels of solution strategies. A new scale named MARS (Maturity of Algebraic Reasoning and Strategies) is proposed to infer maturity levels of algebraic skills. Results from two selected items (Item 1 & 2) which measure the ability of using algebraic equations are reported in this study. Item 3 can be solved by intuitively, whereas Item 7 requires more efforts for "guess and check" method.

Five-item performance-based assessments were administered to 142 US Algebra II students (School 1 and School 2) and 148 Japanese 11th graders (School 3). Students' responses were scored by using the QUASAR holistic rubric (scores raged 0 to 4). Next, solution strategies were all listed and classified into 5 levels on the MARS scale where; 4 is for using algebraic equations, 3 for systematic solution without algebraic equations, 2 for "guess and check," 1 for some understanding, and 0 for no understanding. The MARS scale does not represent the processing ability of the strategies chosen by students. Students' responses were scored on the MARS scale and compared with the QUASAR scores. The heights of the bar diagram in Figure 1 represent the percentage of students scoring at each QUASAR score level. The shading within each of the vertical bars represents the proportion of students using a particular strategies at each QUASAR score level. For example, for School 1, 57% of students scored 4 for Item 1, whereas only 10 % of students scored 4 for Item 2. Utilizing MARS scales, most students who scored 4 used Strategy 2 for both items. Item 1 can be solved intuitively without using equations, but Item 2 requires more efforts to solve it by "guess and check" method. If students used strategy 4 as in other schools, the proportion of students who scored 4 did not change such much.

The results demonstrate that the QUASAR scores do not distinguish the maturity levels of solution strategies. However, the MARS scale reveals students' learning stages at specific QUASAR score levels.

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ASSESSMENT

THE DEVELOPMENT OF AN INSTRUMENT FOR ASSESSING PARENTAL INVOLVEMENT IN STUDENTS' LEARNING OF MATHEMATICS[1]

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A Parent Involvement Questionnaire (PIQ) for assessing parental involvement in students' mathematics learning was developed. It assesses parental roles as motivators, resource providers, monitors, content advisers, and learning counselors. Results of this study suggest that the PIQ may be a reliable and valid instrument for assessing parental involvement in students' mathematics learning. Of the five parental roles, parents as motivators, resource providers, and monitors seem to be the most important predictors of students' success in mathematics. The results of this study also revealed that the students with the most supportive parents as defined by the PIQ not only have higher mathematical proficiency levels, but also more positive attitudes toward mathematics, than those students with the least supportive parents.

The literature on parental involvement in education suggests that parents' involvement benefits students' learning. In fact, the recognition of the important role the parents play in students' education is not new. Hauschmann (1897) indicated that "[a]ll are looking for reform in education....If [the] building is not to be solid, we must look to the foundations--the home" (p. 183). The role of the parents in the education of their children has been continuously recognized to the present time. Researchers have known that the involvement of parents contributes to their children's higher academic achievement, positive attitudes and behaviors, and emotional development (Jacobs, 1993; Weston, 1989). Children of parents who are more involved in school activities perform better than children with parents who are less involved (Stevenson & Baker, 1987; Henderson, 1987). Cross-national studies in mathematics also show a strong relationship between family support and student performance. Stevenson and Lee (1990) suggested that Chinese and Japanese elementary school students' superior mathematical performance might be explained by the fact that Chinese and Japanese parents are more concerned and dedicated to their children's learning than American parents.

Although it is not debatable that parental support facilitates students' learning, the kind of support that is most effective is still an open question. Some researchers (e.g., Nieto, 1992) have indicated that traditional family support (e.g., checking homework) seems not to be as effective as it was expected. Many researchers have attempted to identify the kinds of roles parents may play in students' learning. Previous research in parental involvement focused mainly on two kinds of general involvement: (1) participating in a range of school activities aimed at strengthening the overall school program (e.g., advisory, fundraising, and advocacy activities), and (2) assisting one's own child at home in informal and in school-directed learning tasks (e.g., helping with homework) (Henderson, Marburger, & Ooms, 1986). No research has been done to examine specific kinds of parental involvement aimed at facilitating students' learning of mathematics. The purpose of the study was to develop an instrument for assessing parental involvement in students' learning of mathematics and examine the relationships between parental involvement and students' attitude toward mathematics and mathematical proficiency levels. Given the current broad consensus that students' academic success improves with greater parental involvement, mathematics educators and teachers should promote practices that encourage parental involvement in students' learning of mathematics. The framework for assessing parental involvement in students' learning of mathematics could facilitate such practices.

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[1] Preparation of this paper was supported, in part, by a grant from the Department of Housing and Urban Development (HUD). The authors would like to thank Marie Schimenz and Laurie Rapp for their assistance of data collection. Any opinions expressed herein are those of the authors and do not necessarily represent the views of HUD.

Method

Participants.

The sample consisted of 220 sixth-, seventh-, and eighth-grade students from a public school in a large urban school district and their primary guardians. The students are ethnically and culturally diverse, with 84% of them qualifying for free or reduced lunch. Fifty-seven percent (57%) are African American, 20% are Hispanic, 15% are white, and 8% are "other."

Constructing a Parental Involvement Questionnaire

In this study, parental roles were examined with respect to their support of students' learning in home settings. There are two important dimensions which should be considered in examining parental roles in assisting students' learning in home settings. The first is the type of roles parents play in directly assisting students' learning (e.g., home tutoring, helping homework, forming good learning habits etc.). The second is the type of roles parents play in providing emotional and resource support in students' learning. With this consideration and a systematic review of the literature on parental involvement in early intervention (e.g., White, Taylor, & Moss, 1992), parental belief systems (e.g., Sigel, McGillicuddy-Delisi, & Goodnow, 1992), educational policy (e.g., Sarason, 1995), and mathematics education reforms (e.g., Merttens & Vass, 1993; NCTM, 1989), five parental roles were identified: Motivator, Resource Provider, Monitor, Mathematics Content Adviser, and Mathematics Learning Counselor. Based on these five roles, a Parental Involvement Questionnaire (PIQ) containing 30 items was constructed to assess parental involvement in students' learning of mathematics. When a draft of the questionnaire was developed, two middle school mathematics teachers and seven parents were asked to evaluate each of the items in the PIQ with respect to the relevance, clarity, ratability, and completeness. Revisions were made based on their evaluations. The final version of the PIQ consists of 26 items.

Administration of the PIQ

A subset of the attitude items from the National Assessment of Educational Progress (NAEP) were adopted to measure students' and parents' attitudes toward mathematics. The parental attitude questionnaire and the PIQ were combined into a single instrument and each student was asked to take the instrument home, and have one of their parents or guardians complete it. For each of the items, the parents were asked to decide whether they strongly agree, agree, disagree, or strongly disagree. To "force" tendency selection, no neutral choice was provided. The return rate for the PIQ and the parental attitude questionnaire was 58.6%.

Scoring of the PIQ

Items in the PIQ are statements with positive or negative valences. For each of the statements with positive valences, if a parent or guardian chose "strongly agree", the response was scored as 4 for the item. If a parent chose "agree", the response was scored as 3. If a parent chose "disagree", the response was scored as 2. If a parent chose "strongly disagree", the response was scored as 1. However, for each of the statements with negative valences, if a parent chose "strongly agree", the response was scored as 1 for the item. If a parent chose "agree", the response was scored as 2. If a parent chose "disagree", the response was scored as 3. If a parent chose "strongly disagree", the response was scored as 4.

Measure of Students' Mathematical Proficiency

Students' mathematical proficiency was measured by a mathematical proficiency test which was administered by the school district in the spring of 1996. The test consists of six open-ended, short-answer tasks assessing mathematical knowledge and skills and two extended open-ended problems assessing thinking and problem-solving skills. The scoring was completed by teachers in the school district.

Results

Validity and Reliability

A series of analyses were performed to determine the validity and reliability of the PIQ. Internal consistency (a common measure of reliability) was calculated for the PIQ using Cronbach's coefficient alpha. To establish external validity, correlations were calculated and examined between the PIQ and the parental attitude measure, between the PIQ and the students' attitude measure, and the PIQ and the students' mathematical proficiency levels.

Cronbach's coefficient alpha for the PIQ is .86, which suggests that the PIQ is a reliable measure of overall parental involvement in students' learning of mathematics. The PIQ score is highly correlated with parents' positive attitudes toward mathematics ($r = .82$, $p < .0001$), with the more parental involvement, the more positive the parents' attitude toward mathematics. The PIQ score is moderately correlated with students' attitudes toward mathematics ($r = .24$, $p < .01$) and students' mathematical proficiency level ($r = .22$, $p < .01$).

Parental Involvement and Students' Mathematical Proficiency and Attitudes

Regression analyses suggest that all five parental roles as a whole significantly contribute to predict students' mathematical proficiency levels ($F(5, 119) = 4.02$, $p < .005$). Stepwise regression analyses were also conducted. The results suggest that the independent variables related to parents as motivators, resource providers, and monitors are the most important predictors. Further, the results suggest that independent variables related to parents as content advisers and learning counselors can be removed from the regression model.

Using the PIQ total score, the most supportive and least supportive parents were differentiated. The most supportive parents consist of those parents whose scores are in the top 30% and the least supportive parents consist of those parents whose scores are in the bottom 30%. The mean PIQ score for the most supportive parents is 85, which is significantly higher than that for the least supportive parents, 63 ($t = 17.26$, $p < .001$). Students with the most supportive parents demonstrated higher mathematical proficiency levels than those students with the least supportive parents ($t = 2.38$, $p <.05$). Students with the most supportive parents also demonstrated more positive attitudes toward mathematics than those students with the least supportive parents ($t = 2.16$, $p <.05$).

As was indicated before, the return rate for the PIQ was 58.6%. There are undoubtedly many reasons why 40% of the parents did not complete their PIQ. Whatever the reason, their failure to complete it might be an indication of their reluctance to become involved in their children's learning. If that is the case, a natural hypothesis is that students whose parents completed the PIQ should perform better than those students whose parents did not complete the PIQ. A statistical t-test seems to support the hypothesis. Students whose parents completed the PIQ had significantly higher mathematical proficiency levels than those students whose parents did not complete the PIQ ($t = 3.59$, $p < .001$).

Discussion

The PIQ was developed to assess parental involvement in their students' learning of mathematics. In particular, the PIQ assesses parental roles as motivators, resource providers, monitors, content advisers, and learning counselors. The relatively higher Cronbach's coefficient alpha suggests that the PIQ is a reliable instrument for assessing parental involvement in students' learning of mathematics. With respect to the external validity of the PIQ, the results of this study indicate that the PIQ measure of the parental involvement in students' mathematics learning is highly associated with parental beliefs about mathematics and moderately associated with students' beliefs about mathematics and students' mathematical proficiency levels. Thus, the PIQ may be a reliable and valid instrument for assessing parental involvement in students' learning of mathematics.

Of the five parental roles, parents as motivators, resource providers, and monitors seem to be the most important predictors for students' mathematical proficiency levels. Parents as content advisers and learning counselors are less important predictors. This finding is consistent with the findings from some of the cross-national studies. For example, Stevenson and Lee (1990) found that American parents were more likely to actually help their children with homework than to ask them about their mathematics classes. In contrast, Asian parents were more likely to ask their children about their mathematics classes than to actually help them with their homework. Asian parents (especially mothers) have less formal education than American parents. Therefore, Asian parents may be less able to act as mathematics content advisers and learning counselors than American parents. However, Asian parents consistently motivate their children to achieve academic success and monitor their learning at home. Asian parents' realistic expectations and consistent encouragement might greatly contribute to Asian students' success in mathematics. These findings from the cross-national studies suggest the importance of the parental roles as motivators, resource providers, and monitors in students' learning of mathematics.

The finding that parents as motivators, resource providers, and monitors are the most important predictors for students' mathematical proficiency levels also has practical implications. Given the current mathematics education reform with an emphasis on reasoning, problem-solving, conceptual understanding, and communicating mathematically, school mathematics has changed from the time when parents were in schools. Some new mathematics topics such as discrete mathematics, estimation, applications of calculators and computers have been recently integrated into the school curriculum. As a result, parents might be less able to act as mathematics content advisers and learning counselors at home. In fact, only about one-third of the parents surveyed in this study felt that they knew enough mathematics to help their children at home. In that sense, parents are best able to act as motivators, resource providers, and monitors of students' mathematics learning at home. The teachers, on the other hand, should be expected to take primary responsibility as mathematics content advisers and learning counselors in school.

This study also examined the relationships between parent involvement and students' mathematical proficiency levels and attitudes. The results of this study revealed that the students with the most supportive parents not only have higher proficiency levels, but also more positive attitudes toward mathematics than those students with the least supportive parents. Consistent with studies by other researchers (e.g., Johnson & Jason, 1994; Henderson, 1987), the findings in this study provided empirical evidence that parental involvement benefits students' learning of mathematics. That parental involvement benefits students' learning of mathematics is also evident by the fact that students whose parents completed the PIQ have significantly higher mathematical proficiency levels than those students whose parents did not complete the PIQ.

It should be indicated that parental involvement is neither a necessary nor a sufficient condition in itself for students' academic success in mathematics because parental involvement by itself does not enable students to acquire the full range of mathematical skills and knowledge necessary for success in school (Hoover-Dempsey & Sandler, 1995). Rather, parental involvement is an enhancing variable contributing to students' success in mathematics. Therefore, educators and teachers should promote practices that encourage parental involvement in students' learning of mathematics. Hopefully, the framework and instrument for assessing parental involvement in students' learning of mathematics presented in this paper will facilitate such practices.

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STUDENT PERFORMANCE IN CMP MIDDLE SCHOOL CURRICULUM: ASSESSMENT ALIGNED WITH CURRICULUM

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The Connected Mathematics Project, an NSF-funded middle school curriculum project, has collected data on student achievement using both traditional and non-traditional methods. There are three sources of information being used overall. First, the Iowa Test of Basic Skills Survey Battery Form K mathematics sections (Levels 12, 13 and 14) have been used to help us understand how sixth-, seventh- and eighth-grade students in the Connected Mathematics Project will do on a multiple-choice, timed, standardized test. Second, hour-long tests consisting of items from the Balanced Assessment Project for each of the three grade levels emphasize reasoning, mathematical communication, connections, and problem solving along with a balance of mathematics topics. These tests are aligned with the process goals and curricular topics of the various NCTM *Standards* documents. Third, three 45-minute performance tasks, also developed by Balanced Assessment, have been administered at each grade level. It is hoped that a descriptive analysis of student work on the performance tasks will be useful in understanding the nature of student learning in the CMP curriculum, but this analysis will be discussed in a future paper.

The student populations were of two categories: CMP and Non-CMP. The CMP group consists of students in classes in which the CMP curriculum was to be used throughout the full school year. These classes were drawn from a variety of geographic sites including inner-city urban, urban, suburban and rural. Classes that were not using CMP were selected to form a comparison group that was of similar geographic distribution, similar diversity in the student populations, and similar "tracked" status.

Results

Currently we have administered, scored and analyzed assessment for the sixth and seventh grades. As assessment tools, the ITBS Survey Battery and the Balanced Assessment tests (BA tests) were quite different. The balance of mathematics topics represented was vastly different, with the ITBS being primarily a test about achievement in number concepts, notation and procedures, and the BA tests emphasizing a more even distribution of a wider range of mathematical topics. Another difference was in the nature of the items found on the two tests. The ITBS items focused on isolated concepts, skills, definitions and procedures as can be readily captured in a multiple-choice timed test, while the BA tests emphasized mathematical reasoning, mathematical communication, mathematical problem solving, and connections among mathematical ideas and to real world situations.

Notable in the ITBS Survey Battery results were that all groups performed above the expected grade equivalents by the spring of the year, that growth for each group across all compilations of tests was "average" or slightly better for 7 months of school, and only minimal differences in the growth between the two groups was found. On the BA test, fall scores were similar across grade level groups, but there was noticeable difference in growth over the school year, with the CMP group significantly outperforming the non-CMP group in the spring. The difference in scores was found to be statistically significant for all forms of the test at both grade levels, indicating that students in the CMP curriculum outperformed non-CMP students on items designed to assess topics and goals outlined by the NCTM *Standards* Documents and valued in the CMP curriculum.

COGNITIVE MODALITIES

A CONTEXTUAL INVESTIGATION OF NUMERICAL REASONING AND COMPUTATIONAL ALGORITHMS[1]

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The role of algorithms in the elementary curriculum has been defined on a continuum that ranges from allowing students to invent their own algorithms to teaching traditional algorithms. This paper eschews these extremes and provides an argument for supporting students' construction of personally-meaningful algorithms through the proactive role of the teacher and carefully sequenced instructional activities. Episodes from a third-grade classroom are used to provide a context in which to examine the relationship between conceptual understanding and the use of powerful algorithms. In addition, the episodes also demonstrate how students' beliefs about "school math" influence their understandings of what it means to know and do mathematics.

Introduction

Discussions of the role of algorithms in the elementary curriculum have given rise to conflicting views of their role. These range from encouraging students to invent their own algorithms with minimal guidance to teaching students to perform traditional algorithms. In this paper, we will discuss an approach that eschews both of these extremes. This approach values students' construction of non-standard algorithms. However, it also emphasizes the critical role of the teacher and of instructional activities in supporting the development of students' numerical reasoning. In addition, this approach highlights the importance of discussions in which students justify their mathematical thinking. It therefore frames students' construction of increasingly sophisticated algorithms in conceptual terms and proactively attempts to support their development of increasingly sophisticated numerical interpretations. We bring these aspects to the fore by highlighting the teacher's role in systematically supporting and organizing students' constructive activities, and by focusing on the contribution of instructional sequences designed to support the progressive mathematization of activity.

We outline our viewpoint by presenting episodes taken from a third-grade classroom in which we conducted a nine-week teaching experiment.[2] One of the goals of the experiment was to develop an instructional sequence designed to support third graders' construction of increasingly sophisticated conceptions of place value numeration and increasingly efficient algorithms for adding and subtracting three-digit numbers. Our intent is not to offer examples of exemplary teaching, but instead to provide a context in which to examine the relationship between conceptual understanding and the use of powerful algorithms. In addition, the episodes will illustrate how students' beliefs about mathematics in school influence their development of personally-meaningful algorithms. In the following sections of this paper, we will first outline the instructional sequence developed in the course of the experiment and then focus on the students' algorithms and their beliefs.

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[1] The analysis reported in this paper was supported by the National Science Foundation under grant RED-9353587. The opinions expressed do not necessarily reflect the views of the Foundation.

[2] The authors of this paper were all members of the research team involved in the teaching experiment. The first author was also the classroom teacher who appears in the episodes in this paper.

The Instructional Sequence

As we have noted, a primary goal of the teaching experiment was to develop an instructional sequence designed to support both conceptual understanding and computational facility. The sequence centered around the scenario of a candy factory that initially involved using unifix cubes as substitutes for candies, and later involved the development of ways of symbolizing transactions in the factory (cf. Cobb, Yackel, & Wood, 1992; Bowers, 1995). During initial whole class discussions, the students and teacher negotiated the convention that single pieces of candy were packed into rolls of ten, and ten rolls were packed into boxes of one hundred. Ensuing instructional activities included estimating and quantifying tasks designed to support the development of enumeration strategies.

As the instructional sequence progressed, the students developed drawings and other means of symbolizing as models of their packing and unpacking activity. The goal of subsequent instructional activities was then to support students' efforts to mathematize actual and symbolized transactions in the candy factory so that they might come to signify the composition and decomposition of arithmetical units. To encourage this process, the teacher capitalized on the student's contributions by describing purely numerical explanations in terms of packing activity, and vice versa. In so doing, the teacher also attempted to support the students' development of situation-specific imagery of transactions in the factory that would provide experiential grounding throughout the sequence.

In one type of task posed during this phase of the instructional sequence, students were asked to make drawings to show *different ways* that a given amount of candies might be found in the storeroom if the workers were in the process of packing them. For example, 143 candies might be completely packed up into 1 box, 4 rolls, and 3 pieces, or they might be found as 12 rolls and 23 pieces. When students began describing their *different ways*, they drew separate pictures for each. Later, the teacher encouraged students to use numerals to make a record of their drawing. To this end, she introduced an inventory form that was used in the factory to keep track of transactions in the storeroom. The form consisted of three columns that were headed from left to right, "Boxes," "Rolls" and "Pieces" (See Figure 1). The issue of how to symbolize these *different ways* and thus the composition and decomposition of arithmetical units became an explicit topic of discussion and focus of activity.

In the final phase of the sequence, the inventory form was used to present addition and subtraction tasks in what was, for us, standard vertical column format (See Figure 2).

Boxes	Rolls	Pieces
1	4 ~~14~~ ~~12~~	3 ~~23~~

Figure 1. The inventory form for the candy factory.

Boxes	Rolls	Pieces
3 + 2	2 5	7 8

Figure 2. An addition problem posed on the inventory form.

These problems were cast in terms of Mr. Strawberry, the factory manager, filling orders by taking candies from the storeroom and sending them to shops, or by increasing his inventory as workers made more candies. The different ways in which students conceptualized and symbolized these transactions provided a context in which to discuss their emerging addition and subtraction algorithms.

Classroom Episodes

Throughout the teaching experiment, students engaged in problem-solving tasks that focused on transactions in the candy factory. As the sequence progressed, we inferred that most of the students' activity was grounded in situation-specific imagery of the candy factory. Our primary source of evidence was based on the students' explanations in that drawings and numerals appeared to signify numerical quantities that could be conceptually manipulated. Examples of students' explanations varied, yet personally-meaningful ways of calculating are shown in Figures 3 and 4. We should stress that the non-standard approach of starting with the boxes/hundreds (e.g. Figure 4) was used by many of the students and became an acceptable way to solve tasks. As the goal was *not* to ensure that the students all eventually used the traditional algorithm, the teacher continued to support the development of solutions that could be justified in quantitative terms to other members of the classroom community. Thus, the focus in discussions was on the numerical meaning that students' symbolizations had for them.

It was not until the ninth and final week of the teaching experiment that the issue of *where to start* when calculating emerged as an explicit topic of conversation. The problem posed was: *There are five boxes, two rolls, and seven pieces in the storeroom. Mr. Strawberry sends out an order for one box, four rolls and two pieces. What is left in the storeroom?* Aniquia offered the first solution as shown in Figure 3.

Boxes	Rolls	Pieces
4 ~~5~~ - 1	12 ~~2~~ 4	7 2
3	8	5

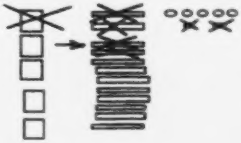

Figure 3. Aniquia's solution.

Boxes	Rolls	Pieces
5 - 1	12 ~~2~~ 4	7 2
~~4~~ 3	8	5

Figure 4. Bob's solution.

She explained that she first took two pieces from the seven pieces in the storeroom and that she then unpacked a box so that she could send out four rolls. The teacher drew the graphics of boxes, rolls, and pieces to help other students understand Aniquia's reasoning (See graphics in Figure 3). After asking if there were questions for Aniquia, the teacher posed the following question to the class, "Did anybody do it a different way?" Bob raised his hand and shared his solution method, which involved sending out a box first (See Figure 4). After Bob had finished his explanation, the teacher asked:

Teacher: All right. Now Bob said I'm gonna send out my box before I send out my rolls and Aniquia said I'm gonna send out my rolls before I send out my box. Does it matter?

The students' responses included:

Cary: t depends on the kind of problem it is.
Avery: Nope, you can do it either way.
Anita: When you usually do subtraction you always start at the right 'cause the pieces are like the ones.
Avery: But it doesn't matter.
Rick: Yeah, 'cause we're in the candy factory, not in the usual stuff.

At the end of this session, the project team reflected on the students' discussions about *where to start* and agreed that it would be productive to revisit the issue. We felt that many of the students' justifications derived from their attempt to recall the standard algorithm. The students were third graders and had received two years of traditional instruction during which they were taught the standard algorithms for adding and subtracting two-digit numbers. It appeared from the students' conversations that they agreed that it did not matter where they started when they were acting in the candy factory rather than doing "the usual stuff." We speculated that Cary's comment about depending on the kind of problem referred to whether or not it was posed in the context of the candy factory. For most students, there appeared to be two different "maths" — regular school math and the math they did with us in the setting of the candy factory. They were therefore able to justify the difference in *where to start* in terms of the different obligations and expectations in the two instructional situations. In particular, the students seemed to realize that there were crucial differences in what constituted acceptable explanations and justifications in each situation. Explanations in the candy factory focused on imagined quantitative transactions in the storeroom, and on how they could be symbolized on the inventory form. School math entailed explanations of symbol manipulations that did not have to be justified in quantitative terms.

The next day, the first problem posed to the class was *There are three boxes, three rolls and four pieces of candy in the store room. Mr. Strawberry gets an order for two boxes, four rolls, and one piece. How many candies are left in the store room after he sends out* *his order?* The first solution was offered by Martin, who completed the task by first sending out the boxes. The next solution offered entailed first sending out the pieces (See *Figure 5*). *At this point*, the teacher decided to capitalize on these two different solutions by reintroducing the question of *where to start*.

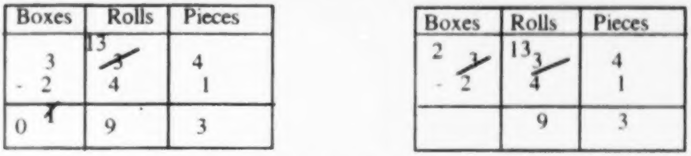

Figure 5. Two subtraction solutions.

Teacher: Does it matter? Is it okay either way?
Rick: No.
Teacher: But we get the same answer.
Rick: But it's confusing. It's harder to work it out. It's easier that way (points to form on the right).
Teacher: But my question is do we have to do pieces, then rolls, then boxes or can we do the boxes first.
Bob: No.
Avery: You can start with boxes or rolls.
Rick: Yeah, but that's confusing.
Teacher: Okay, let me tell you what I'm hearing you say. I'm hearing you say that it's easier and less confusing [to start with pieces].
Anita: I think you can start with boxes or pieces but not the rolls.
Bob: It would be harder to start with rolls.
Avery: Yeah.

The focus of the discussion shifted during this exchange to the issue of which symbolized solution process was *easier* or *less confusing* to understand. The students seemed to agree that the two processes would result in the same solution; their emphasis was on ease of comprehension. We should stress that the teacher's questions were not intended to steer the students to the standard algorithm. Instead, the students' judgments of which approach was easier to understand was made against the background of their prior experiences of doing mathematics in school.

After these discussions, we posed tasks in the traditional textbook column format without the inventory form.

Teacher: This is kinda different from the inventory form.
Jess: It's kinda different 'cause it doesn't say boxes, rolls, pieces.
D'Metrius: You still don't have to have boxes, rolls, pieces. It don't have to be up there 'cause you know these are pieces, rolls, and boxes.
Jess: I just know that. . . I remember boxes, rolls, and pieces help me.

It appeared from the conversation that the students evoked imagery of the candy factory as necessary to support their reasoning. This conjecture was reinforced by the observation that several of the less conceptually advanced students, working in pairs on activity sheets, continued to draw boxes, rolls, and pieces to support their activity. In addition, they produced a variety of different solutions. These included adding from the left, adding from the right, and working from a drawing and recording the result.

The episodes discussed in this article provide support for a change in emphasis "toward conjecturing, inventing, and problem solving and away from an emphasis on mechanic answer-finding" (National Council of Teachers of Mathematics, 1991, p. 3). This should in no way be construed to mean that students do not need to construct powerful algorithms; it simply calls attention to the importance of students' developing increasingly sophisticated numerical concepts as they develop personally-meaningful algorithms.

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RELATIONSHIP OF AFFECTIVE AND COGNITIVE ASPECTS OF LEARNING

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Introduction. The NCTM *Standards* (1989) and recent research on affective aspects of learning (McLeod, 1992) highlight the importance of students' dispositions to mathematics learning, while constructivist perspectives focus on cognitive growth. This study indicates children's cognitive growth as they conceptualize more powerful ways to solve mathematical tasks, bringing about affective gains by increasing their sense of mathematical control. When facing a task, the child's conceptions serve in setting goals and implementing activities that are included in these conceptions. If the activities do not solve the task, the child faces a problem and may change the activities, i.e., modify the conception, until a solution is reached. Repeating the modified activities to accomplish similar goals establishes the modified conception, enhances the child's sense of control, and produces pleasant emotions.

Methodology. A teacher-researcher worked in the computer microworld Sticks with two fourth graders to study their construction of fraction knowledge. The analysis focused on the children's affective experiences as they modified their fraction schemes.

Analysis and Interpretations. Initially, Linda and Jordan solved a task of sharing a candy stick (among two and three people) using the activities of estimating a point for a cut, breaking the stick, and comparing *all parts*. To estimate "one third," they swept the cursor from left to right in a way that indicated iteration of the parts, albeit implicitly. Linda's comment, "This might take a long time," indicated the problem they faced due to the cumbersome work involved. In the next task (four people), building on the implicit use of iteration, the researcher challenged them: "Could you use only one part?" In response, the children re-organized their activities by including explicit use of iteration. They repeatedly took turns in estimating one part, iterating it four times, then comparing the composed stick to the original—a new conception and sequence of activities that we termed the *equi-partitioning scheme*. The children's actions and language indicated a high sense of control and excitement, as they were re-adjusting the size of one person's part to accomplish their goal—a candy stick composed of four equal parts that has the same size as the original stick.

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BEYOND COMPUTATIONAL SKILLS AS ROTE ACTIONS: THE IMPORTANCE OF THE CONTEXTUAL AND CONCEPTUAL ASPECTS OF ALGORITHMIC PERFORMANCE

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Current school mathematics has shifted its focus from traditional algorithms to problem solving and mathematical understanding (Hiebert & Carpenter, 1992; NCTM, 1989). Consequently, a variety of suggestions from different perspectives have been proposed about whether and how formal algorithms need to be taught in elementary math classrooms (Kamii, 1994). With an awareness of the necessity of clarifying the relationship among "computational skills", "algorithmic performance", and "problem-solving competence", it is the purpose of this study to conceptualize "algorithmic performance" and to illustrate the importance of its different aspects empirically.

Algorithmic performance can be conceptualized as including three aspects, that is, (1) its contextual aspect, (2) its conceptual aspect, and (3) computational actions. The contextual aspect of algorithmic performance accounts for why a computational action can be taken in a specific condition, and the conceptual aspect of algorithmic performance may explain how a computational action can be carried out. Isolated computational skills are rote actions without concern for conditions and without understanding. Differently, identifying and emphasizing the contextual and conceptual aspects of algorithmic performance let us treat algorithmic performance as a problem-solving competence. This conceptualization serves as the theoretical base for this study in the empirical investigation.

Sixth graders' (N=26) algorithmic performance was investigated with three specifically designed tests. Comparatively, the students' computational skills were the best aspect among all of the aspects being tested. However, these students' acquired computational skills may be rote actions. Their poor performance shown on the contextual and conceptual aspects of algorithmic performance implied that these students' algorithmic performance was limited to computational skills. Without realizing and emphasizing on the contextual and conceptual aspects of algorithmic performance, it was difficult for these students to get the potential benefits from teaching and learning algorithms. Consequently, developing students' problem-solving competence deviated from instruction on algorithms.

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EXPLORING THE EFFECTS OF CONCRETE OBJECTS ON SOLUTION PROCESSES AND ARITHMETIC WORD PROBLEMS' DIFFICULTY

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With research on addition and subtraction word problems, several process models have been proposed to explain why some word problems are more difficult than others (e.g., Briars & Larkin, 1984). It has been revealed that a "compare" problem is the most difficult type of one-step word problem (e.g., Briars & Larkin, 1984; Stern, 1993). However, there are less efforts to address whether and how the difficulty of a problem may be changed in a setting with different concrete objects.

In a previous study (Li, 1994), a third grader's performance exhibited that the difficulty of a word problem might be changed if some specific concrete objects were provided. Rather than representing 'numbers' stated in a compare problem with provided concrete objects (no puppets provided), the third grader's own occasional drawing of personal models helped her to re-represent the compare problem and to solve the problem successfully. This result indicated the need for a further investigation. Specifically, this study extended the previous study in two ways: (1) to examine whether and how other third graders' understanding of word problems would be affected by the provided concrete objects with some puppets; and (2) to explore what might affect problem difficulty in a setting with the provided concrete objects.

Fourteen third graders were individually interviewed in this study. Results from this study suggest that: (1) The efficient use of provided concrete objects for understanding problems needs to be taught to students explicitly. Without such experiences, the provided concrete objects with a problem would not facilitate students' understanding of the problem in a way as educator/researcher expected. (2) With no clear indication of the effects of provided concrete objects on problem difficulty, the difficulty of used word problems was differentiated by the characters of these problems' own contextual factors. Comparatively, compare problems were the most difficult type of word problems that confirmed other studies' results. The semantic features of compare problems had the main contribution for the noted problem's difficulty. Furthermore, a compare problem that seemed to allow different mathematical modeling could lead students to make more mistakes in their solutions.

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COGNITIVE PROCESSES IN SOLVING TWO-STEP ADDITIVE COMPARE PROBLEMS

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Children's difficulties in solving one-step compare problems are likely to occur when the structure of the presented information is inconsistent with the order of operation required of the problem (Verschaffel, 1994; Stern, 1993). This study extended one-step compare problems to two-step compare problems to understand if children have a preference for the correspondence between the order of problems presentation and the order of operations required of the problem.

The experiment included three types of problems: consistent language (CL), partially inconsistent language (PIL), and totally inconsistent language (TIL) structures. Each problem contained four sentences: one assignment, two relational sentences including relational terms such as "more than" or "less than", and one question. Thirty fourth graders of Taiwan solved individually two sets of eight compare problems requiring the combination of operations (+ &-). For example,

> Joe has 4 marbles. (assignment sentence) Tomhas 5 marbles more than Joe. (relational sentence) Pat has 6 marbles less than Tom. (relational sentence) How many marbles does Tom have? (question sentence).

Each subject was asked to retell the problems based on his or her solution and the cues on the cards. Retelling was important to provide a record of children's thinking and a record of their ability to reconstruct the structure of the problems.

Results indicated that children have more difficulties on problems with more inconsistent language structures in relational sentences. The problems requiring the operations (-, +) with CL structure were significantly easier than the problems with PIL and TIL structures, $F(2, 58)=6.23$, $P<.05$. Means of CL, PIL, TIL are .92, .67, .57, respectively. The problems requiring (+, -) with TIL structure were significantly more difficult than the problems with PIL structure, $F(2, 58)=23.88$, $P<.05$. Means are .98, .67, .23, respectively. Children had a tendency to retell problems using consistent language structure. Children's unsuccessful conversions from PIL and TIL into CL problems occurred during the phase of problem representation transformation into mathematical expression rather than vice versa. The incorrect retellings with two changed relational sentences frequently occurred in TIL problems and the retellings with one changed relational sentence frequently occurred in PIL problems. The findings yield strong evidence in favor of the "consistent hypothesis" (Lewis & Mayer, 1987).

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CURRICULUM REFORM

ORIGINS AND IMPACT OF THE NCTM *STANDARDS*: THE VIEWS OF THE DEVELOPERS

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Leaders of the National Council of Teachers of Mathematics (NCTM) and writers of the *Curriculum and Evaluation Standards for School Mathematics* were surveyed to determine their views of the major forces that influenced the development of the *Standards*. They were also asked to estimate the impact of the *Standards* on various groups. Questionnaire data were supplemented with extensive interviews. The results suggest that the *Standards* were influenced most heavily by forces from within mathematics education, rather than by publications like *A Nation at Risk* that came from outside the field. Developers thought that the *Standards* had made an impact on educational policymakers, but that NCTM had been less successful in promoting change in classrooms.

The publication of the *Curriculum and Evaluation Standards for School Mathematics* (NCTM, 1989) was a significant event in mathematics education in North America. Although the *Standards* have not yet been implemented in many schools, NCTM has had an important impact on mathematics education policy at both the national and state levels. In this paper we discuss the origins and impact of the *Standards*, relying mainly on data gathered from the developers of the *Standards* through questionnaires and interviews.

Our data were gathered as part of a larger case study of educational change in mathematics (McLeod, Stake, Schappelle, Mellissinos, & Gierl, in press). That case study focused on understanding the origins of the NCTM *Standards*, as well as their development, dissemination, and impact in K-12 classrooms and in the educational community in general. Main sources of data included interviews with NCTM leaders, such as past NCTM presidents and those involved in writing the NCTM *Standards*. The data reported in this paper come mainly from questionnaires submitted to the nine members of the NCTM Commission on Standards for School Mathematics and to the 24 writers of the *Standards* (see appendix). Responses were obtained from six members of the Commission and 18 writers, a response rate of 73%. We also include some data from interviews with other educational leaders.

Origins of the NCTM *Standards*

Although the first sentence of the NCTM *Standards* cites *A Nation at Risk* (NCEE, 1983), and that provocative document is thought by some to be the most important source of the *Standards*, our respondents rated the NCTM's *Agenda for Action* (NCTM, 1980) and the California *Mathematics Framework* (California Department of Education, 1985) as more important influences. (See Table 1.) Other documents, like the reports of the "Options for the 1990s" conference (Romberg, 1984) and the "New Goals" conference organized by CBMS (1984), were viewed as very important influences only by the members of the Commission. Many members of the Commission had attended those

meetings and knew the influence of those conferences on the events leading to the NCTM *Standards*.

Table 1. Ratings of Documents That Influenced the NCTM Standards
(Not important = 1, very important = 4)

Document	Commission (n = 6)	Writers (n = 18)	Total (n = 24)
An Agenda for Action (1980)	3.67	3.28	3.38
A Nation at Risk (1983)	3.00	3.06	3.04
School Mathematics: Options for the 1990s (1984)	3.42	2.82	3.00
New Goals for Mathematical Sciences Education (1984)	3.20	2.32	2.55
The California *Mathematics Framework* (1985)	3.40	3.06	3.13

The members of the Commission often described the origins of the *Standards* in terms of the important meetings mentioned above. As one leader put it, "What I specifically remember--which from my point of view was a fairly dramatic event--was the wrap-up session of the [New Goals] Conference (CBMS, 1984). At that session, Joe Crosswhite introduced two motions that led to the *Standards* and to MSEB." The following month the "Options for the 1990s" meeting at Wisconsin (Romberg, 1984) discussed the same issues and made some similar recommendations. Although other factors were also instrumental in promoting the notion of "standards," these two meetings were central.

The writers, however, had not played a central role in those meetings. Given their task, the writers naturally looked at other statements of curricular goals, including curriculum guidelines from California, Oregon, Wisconsin, and other states. The California Framework (California Department of Education, 1985) was mentioned frequently. As one writer noted, "Certainly the 1985 California Framework was one of the documents that was used in helping to formulate the NCTM *Standards*. It was something that everybody in all of the groups was familiar with and looked at for help in thinking about what the *Standards* might contain." California is the most populous state, and Californians are often concerned that their large numbers within NCTM are resented. Some Californians suggested that they have to be careful not to push too hard since people will reject an idea "just because it comes from California." But our sources did think that the reform effort in California was a major influence; for example, John Dossey noted that "The California Framework of 1985 talked about mathematical power," an idea that became "a central theme for the *Standards*."

Many other sources of the *Standards* came up in interviews and are described in a longer report (McLeod et al., in press). For example, President Shirley Hill (1981) outlined the issues that NCTM needed to address in providing professional leadership in mathematics education, noting that the 1980s could be "a decade for mathematics" (p. 10). In early 1983 NCTM's Instructional Issues Advisory Committee was assigned the task of developing standards for textbook selection, and their thinking gradually expanded to include standards for curriculum, instruction, and evaluation. The Research Advisory Committee of NCTM was asked to evaluate John Saxon's textbooks in 1983, and the committee noted that they would need "standards" to evaluate curriculum materials. Around the same time, NCTM was trying to determine the effectiveness of the 1980 *Agenda*, and it became clear that a more detailed specification of curriculum guidelines would be useful. These concerns led NCTM to seek funding for the development of the *Standards;* however, their grant proposals were unsuccessful and NCTM decided to pay for the *Standards* out of their own funds.

Most NCTM leaders take great pride in the fact that the NCTM *Standards* were developed without funding from the federal government. Although the US Department of Education provided funds for the "Options for the 1990s" conference (Romberg, 1984) and NSF funded the "New Goals" meeting (CBMS, 1984), only one of our 24 respondents thought that these federal agencies had a major influence on the development of the NCTM *Standards*. (See Table 2.) This view was very different from the position of the federal officials that we interviewed, who thought that NSF and Department of Education support had been crucial to NCTM.

Table 2. Organizations Rated as Major Influences on the Development of the Standards

Organization	Commission (n = 6)	Writers (n = 18)	Total (n = 24)
National Science Foundation (NSF)	0%	6%	4%
Office of Educational Research and Improvement (OERI)	0%	0%	0%
Mathematical Sciences Education Board (MSEB)	67%	56%	58%
Second International Mathematics Study (SIMS)	50%	17%	25%

Of course, the *Teaching Standards* (NCTM, 1991) were funded in part by NSF, and so was the review of the draft of the *Standards* that was conducted by MSEB (1988). This review obtained feedback from several different "focus groups" (including parents, mathematicians, scientists, and others) to help NCTM revise and build support for the *Standards*. As a Commission member noted, support for the 1989 *Standards* was growing at this time and "NSF was almost inviting us to come for money"; however, what most Commission members seemed to remember was the initial resistance of federal officials to the idea that NCTM should develop "standards."

The Commission members and writers noted the important role that MSEB played in supporting the development of the *Standards*, and the Commission members were well aware of the importance of the SIMS data (Table 2) and their impact on policymakers. The writers of the *Standards* were not as inclined to see the SIMS data as a central influence, and in fact the SIMS report was not cited in the 1989 document. Most of the leaders and writers came to agree that posing the *Standards* as a response to the SIMS data would be "reactive rather than proactive." Instead, the consensus was that the *Standards* should be "a proactive, positive statement; let's say what we believe and then act on it."

Contextual and Political Influences on the NCTM *Standards*

During the 1980s one common interpretation of the term "standards" emphasized the notion of accountability; an agency sets "standards" which are then used to evaluate students, teachers, or schools. The early discussions of the NCTM *Standards* emphasized accountability issues, but the writers were influential in changing that emphasis to a focus on the NCTM *Standards* as a vision of what school mathematics should be. The difference between the Commission and the writers on the importance of accountability issues appeared in our questionnaire data, where only 39% of the writers thought that accountability was a major influence. (See Table 3.) Members of the Commission, with their emphasis on policy issues and their recollections of the origins of the *Standards*, rated accountability as a major influence much more frequently than the writers. Other factors that influenced the development of the *Standards* included technology, equity, and changes in learning theory, especially the emphasis on constructivist ideas about learning. More general issues, such as education for global competitiveness and an increased emphasis on teacher professionalism, were an important part of the rhetoric around the *Standards*, but neither the Commission nor the writers rated those issues very highly.

Table 3. Important Contextual Factors that Influenced the Development of the Standards (percent responding "major influence")

Factor	Commission (n = 6)	Writers (n = 18)	Total (n = 24)
Accountability issues (standards as quality criteria)	83%	39%	50%
Improvement of classroom instruction (standards as a vision)	100%	78%	83%
Improvement of classroom assessment	83%	67%	71%
Equity issues	83%	61%	67%
Need to update mathematical content of the curriculum	83%	83%	83%
New views in the philosophy of mathematics	83%	72%	75%
Emphasis on applications of mathematics	100%	72%	79%
Education for economic advancement and global competitiveness	33%	6%	13%
Changes in technological tools	100%	78%	83%
Changes in learning theory to cognitive psychology, to constructivist views of learners, or to holistic approaches	83%	61%	67%
Movements to empower teachers and to increase teacher professionalism	33%	44%	42%

Judgments of Impact

The impact of the NCTM *Standards* has been substantial in some areas, but change in the schools is occurring only very slowly. Our sources agreed that the amount of change that has occurred in classrooms declines as one moves from elementary to middle to high school (see Table 4), an estimate that is supported by data from a national survey (Weiss, Matti, & Smith, 1994). The decline was judged complete at the college level, where the

Commission and the writers agreed that the *Standards* had not had a major influence on mathematics teaching. The Commission members were often more optimistic about the extent of the influence of the *Standards* than the writers, especially in the case of K-12 teachers; data from Weiss et al. (1994) suggest that the writers perceptions are more accurate. Also, the writers do not believe that the *Standards* are having a major impact on teacher education institutions. If the writers' concern is justified, a concerted effort at disseminating the *Standards* to teacher educators should be a priority. (See MSEB, 1996, for a discussion of teacher education issues.)

Table 4. Judgments about the Influence of the Standards on Various Groups
(percent responding "major influence")

Groups	Commission (n = 6)	Writers (n = 18)	Total (n = 24)
Educational policy makers	83%	22%	38%
Textbook publishers	67%	61%	63%
Test publishers	50%	11%	21%
Teacher education institutions	50%	22%	29%
Federal agencies	67%	33%	42%
State agencies	50%	44%	46%
Political leaders	33%	6%	13%
K-4 mathematics teachers	83%	22%	38%
5-8 mathematics teachers	67%	17%	29%
9-12 mathematics teachers	50%	6%	17%
College mathematics teachers	0%	0%	0%
Other curriculum areas	0%	17%	13%

At the policy level, Blank and Pechman (1995) report that 41 states have changed their curriculum guidelines for mathematics to bring them more closely in line with the NCTM *Standards*. The writers seemed less aware of these substantial changes at the policy level than were the members of the Commission. Clearly the NCTM *Standards* have had a major impact on policymakers, especially those at the federal and state levels who decreed that all the other school subjects needed standards as well. The *Standards* have also had an influence on textbook publishers, according to our sources, so it may be reasonable to hope that the influence on classrooms will be greater in the future.

In summary, the *Standards* have apparently had a substantial impact on policymakers at the state and national level. The impact on classrooms has been relatively slight so far. A major concern, however, is whether the *Standards* can withstand the current assault from politicians and parent groups who believe traditional mathematics, with its emphasis on rules and procedures, is the heart of school mathematics.

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Appendix

Members of the Commission on Standards for School Mathematics included Tom Romberg, Chair; NCTM presidents Iris Carl, Joe Crosswhite, John Dossey, Shirley Frye, and Shirley Hill; James Gates, NCTM Executive Director; Dale Seymour, a publisher; and Lynn A. Steen, Mathematical Association of America.

The four chairs of the working groups were Paul Trafton (K-4), Glenda Lappan (5-8), Christian Hirsch (9-12), and Norm Webb (Evaluation). Although these four were also members of the Commission, we classified them in our data analysis as writers.

In addition to the four chairs of the working groups, the writers included Hilde Howden, Mary Lindquist, Ed Rathmell, Thomas Rowan, and Charles Thompson for K-4; Dan Dolan, Joan Hall, Tom Kieren, Judith Mumme, and James Schultz for 5-8; Sue Ann McGraw, Gerald Rising, Hal Schoen, Cathy Seeley, and Bert Waits for 9-12; and Elizabeth Badger, Diane Briars, Tom Cooney, Tej Pandey, and Alba Thompson for Evaluation.

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RESEARCH ON INTERDISCIPLINARY TEACHING AND LEARNING

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One of the goals of the reform movement in mathematics education is to encourage interdisciplinary teaching. Organizations such as National Council of Teachers of Mathematics, National Science Teachers Association, Mathematics Association of America, and School Science and Mathematics Association have supported such efforts. Classroom practice is beginning to reflect this initiative, particularly in the integration of mathematics and language arts as well as mathematics and science.

Currently, there is little research to provide the theoretical framework necessary to lead these efforts. There are many questions to be answered about the interdisciplinary approach to learning. Some of these include:

1. What are the perceptions of usefulness of interdisciplinary learning among teachers, preservice teachers, administrators, students, and teacher education faculty?
2. How is achievement affected by integration efforts?
3. What affect does this interdisciplinary approach have on students' abilities as problem solvers and problem creators?
4. What affect does this approach have on students' critical thinking skills?
5. What are actual classroom practices?
6. What role does and should technology play?
7. How does this issue address diversity?
8. How should we look at assessment?

The participants in this session will describe current research goals as well as important directions that research in this area should take. It is anticipated that the participants will be those currently involved in interdisciplinary learning and research as well as those who have an interest in this effort.

Session Format:

1. (40 minutes) Panel members, representing a variety of integration efforts, will discuss current practice and research.
2. (40 minutes) Participants will be invited to share their ideas about the direction of research that is needed to provide a theoretical framework.
3. (10 minutes) This time will be provided so that participants can organize into groups of research interests for future contact. A directory of individuals will be compiled from these groups.

PME, as indicated by the goal statements, is actively interested in interdisciplinary research as well as the implications of teaching and learning mathematics. Integration of mathematics with other areas is an important endeavor in which research is currently inadequate. This discussion group is an attempt to provide a better understanding of the assumptions and ramifications of interdisciplinary teaching and learning.

DOCUMENTING SYSTEMIC REFORM IN ONE MIDDLE SCHOOL

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Systemic Reform: *What* is it, and how can it be documented? Grounded in the contructivist theory of learning, systemic reform reconceptualizes school reform by combining bottom-up reform with top-down support (Darling-Hammond, 1994). The purpose of this research project was to use classroom change indicators developed by the NCTM and a set of systemic reform indicators developed by the Virginia State Systemic Initiative to document reform in one middle school. The Lead Teacher Component played a major role in this initiative, and the middle school selected for this 1996 indepth case study participated in Virginia's NSF-funded State Systemic Initiative and has had two trained lead teachers in place since the summer of 1993.

The complexity and breadth of systemic reform required that data collection not be limited to the school. Artifacts, interviews and observations were used selectively to collect *post hoc* longitudinal data from four different levels: (1) classroom, (2) building, (3) district, and (4) state. Three particularly significant areas of consensus emerged among these four levels: It is important to (1) change the way mathematics is taught, (2) encourage professionals at all levels to participate in dialog and activities related to this change, and (3) facilitate professional collaboration. These were supported by the following data:

- The state's newly revised Mathematics Standards of Learning (SOLs) are closely aligned with the NCTM *Standards*.
- The school district joined a local consortium of counties to work together on providing inservice and professional support for activities related to the implementation of these SOLs. (Lead teachers from the middle school participated in all of these functions)
- A task force made of up teachers and administrators was created to rewrite and redesign the county's mathematics curriculum. Two points stand out as significant about this activity: (1) teachers were given an opportunity to have input, and (2) for the first time teachers were paid for the many hours they spent working during the summer.

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EVALUATING MODELS OF PRACTICE: REFORM-BASED MATHEMATICS AT THE MIDDLE SCHOOL LEVEL

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There is a great need to critically evaluate proposed models of the influences of teachers' instructional practices. This evaluation is done within a larger project[1] seeking an empirical base of research on teaching early algebra skills that can identify factors that influence middle school math teachers' instructional practices and student learning, and that ultimately support a generative model for effective instructional innovation.

We evaluated the utility of a general class of models that describe the influences of teachers' beliefs and knowledge on teachers' instructional practices using data from a sixth grade math teacher (Anne) during a discussion on how to solve a Problem of the Week. Video tapes of the teacher's and students' actions and speech are supplemented with interview data and samples of students' homeworks and assessments. Teacher models tend to highlight three areas of (potentially overlapping) teachers' knowledge: Views of how students learn mathematics, views of the domain of mathematics, and teachers' pedagogical content knowledge. From Alba Thompson's (1992) review, one can infer a causal model where each of these *funds of knowledge* directly influences teacher practices.

We evaluated the utility of the model as a means for organizing our data. Several important limitations arose. First, student performances strongly influenced Anne's instructional practice and later planning. For this reason we found it necessary to add to the current model a feedback loop from student learning performance back to the teacher's instructional practice, as the teacher re-evaluated her practice based on formative assessments of student performances. Discourse analyses also revealed an arbitrary delineation of various knowledge domains which are confining and vague. The influences of various funds of knowledge on Anne's practice are quite interactive and mutually shape her instruction. We found it necessary to view these influences as partially overlapping and dynamic in order to capture their role in shaping Anne's practice. Our investigation also revealed a need to explicitly include the impact of high-stakes regional assessments (such as the CAT) as a factor influencing instructional practices. We provide empirical support for links among components that have received limited discussion in the literature. Our inquiry suggests that evaluatiuon of such models is valuable because it explicitly links empirical and theoretical efforts aimed at understanding teachers' practices. It also points out limitations inherent in the generation and evaluation of models on a limited set of data.

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THE CONNECTED MATHEMATICS PROJECT (CMP)

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CMP is a new mathematics curriculum for grades six, seven, and eight. The curriculum is in units, eight per year, dealing with big ideas in number, geometry, measurement, statistics, probability, and algebra. All of the curriculum is presented in large problems in the contexts of real situations, mathematical situations, or whimsical situations. Problems are presented to the students using a problem solving model of launching, exploring, and summarizing. Students are encouraged to discuss their ideas with their peers and with their teacher. Through student discussions, the main ideas are sorted out and made more clear to all of the students. Each investigation ends with a section called Mathematical Reflections. The students are asked frequently to write about their ideas as part of their reflections. Careful assessment has been developed using many different schemes, including checkups, partner quizzes, unit tests, projects, self checks, and journals.

Two major studies of student achievement have been conducted. The first compared 1000 sixth and seventh grade CMP students with 500 non-CMP students on pre and post tests of the Iowa Test of Basic Skills, and on a Balanced Assessment Open Ended Problem Solving Test[1]. In that study, the CMP students performed as well as non-CMP students on the ITBS, and outperformed significantly the non-CMP students on the Balanced Assessment Test. In the second study, 180 CMP seventh grade students were compared with 100 non-CMP students on a battery of proportional reasoning tasks dealing with rates, ratios, and scaling[2]. On this test, the CMP students outperformed the non-CMP students by a ratio of about 2 to 1. Both studies have been continued with eighth grade students. The results are not yet complete. Copies of both studies are available here at the Poster Session, and from CMP.

The video shows Mary Bouck teaching a ratio problem (orange juice) to a group of sixth grade students. This problem has since been moved to the seventh grade, now included in a unit called *Comparing and Scaling*.

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and Eric Knuth. Special thanks (!) to Amy French

[1] Zowajewski, Judy. Student Performance on ITBS Survey Battery and Balance Assessment Tests Grade 6 and 7, 1994-95 School Year., Draft 6/28/96

[2] Ben Chaim, David, et al. Proportional Reasoning of Seventh Grade Students, Connected Mathematics Project vs. Traditional Curriculum Draft

DISCOURSE

DEVELOPING FORMAL MATHEMATICAL LANGUAGE STRUCTURES IN MIDDLE SCHOOL*

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This paper reports classroom data collected over a two year period which suggests that the teacher's examination of her belief structure and development in the content area resulted in a change in the language she used in the classroom. In addition, evidence indicated that this change influenced the students' willingness to believe in the content. In particular, this paper explores the students' understanding and usage of the logical if-then structure in both the presence and the absence of explicit mathematical language discourse. Students voluntarily used the if-then structure in everyday discourse and demonstrated an increased understanding of the contrapositive. This finding surfaced when data from the two years were compared: 26% of the students in year one had a understanding of the contrapositive compared to 50% in the class in year two of the study.

Introduction

The teacher of mathematics must have knowledge not only of the subject matter, but its nature and discourse as well. Ball (1991) referred to these as knowledge of mathematics and knowledge about mathematics, respectively. This type of discourse plays a significant role in the classroom envisioned by the Professional Standards (National Council of Teachers of Mathematics, 1991). However, its implementation remains somewhat a mystery in everyday classroom activity. The dilemma becomes how to move "regular" classroom discourse, consisting of everyday language, in the direction of mathematical discourse that supports the building of mathematical language and content knowledge. The teacher's role in this process is critical, as she or he must guide the student through the maze where they first accept that talking is good, then move them into talking mathematics using everyday language, and finally to move the student to everyday talking using mathematical language.

This process requires a rethinking of the roles of the teacher and the students. Cobb (1988) maintains that "One of the teacher's primary responsibilities is to facilitate profound cognitive structuring and conceptual reorganizations" (p.89). When a teacher undertakes a facilitative role in classroom discourse, students not only express their mathematical thinking but also conceptualize situations in a variety of ways (Yackel, Cobb, Wood, & Merkel, 1990). This role requires the teacher's knowledge to expand beyond just the subject matter to include the nature of mathematics as well as its discourse. It is the exploration of one's beliefs, a reflective process, that allows the individual to evaluate their own knowledge. Sowder, Philipp, Flores, and Schappelle (1995) indicate "One's

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knowledge about mathematics is intricately interwoven with one's beliefs about the nature of mathematics itself" (p. 256).

Our research is beginning to reveal the complexity of the implementation of this process in the middle school classroom (Price & Adams, 1995; Adams & Price, 1995).
In year one of this study, the teacher and students investigated a mathematical rule introduced by a student relative to divisibility by eight. The original version of the 2-rule was stated as: If you can divide a number by 2 three times evenly, then the number is divisible by eight. The teacher facilitated the student development of the content by allowing a total of 32 minutes of whole-class discourse over a three day period. There were 78 recorded interactions contributed by the 14 students participating in the discussion.

Data revealed that the teacher was also an active participant in the construction of the rule's validity. In addition, complexities of the mathematical if-then structure relative to the idea of a rule "working" surfaced and was experienced by the teacher as well as the students. The students in this 6th period did not appear to understand that the mathematical content of the 2-rule necessarily implied that the converse, the inverse, and the contrapositive were all valid mathematical statements. One of the factors contributing to the students lack of knowledge of and/or about mathematics was the lack of experience with mathematical language necessary to support the development of the content.

We held summer meetings with the teacher at the end of year one in which the data relative to the classroom episodes dealing with the 2-rule were reviewed. As the teacher viewed the video tapes of her classroom, she reflected on both the interactions and the instruction. This activity allowed time for the teacher to reflect not only on the discourse relative to the content but on her understandings relative to the implications of the rule as well. Based on the continuing analysis, a document designed to address knowledge of and about mathematics through explicit attention to the structure of the if-then implication statement and the mathematical language needed to support the development of that content was prepared by the researchers. In particular, the document explored various structural forms of the if-then statement in the context of division by two, providing the mathematical language that supported the multiple ways in which these relationships could be discussed.

The document was given to the teacher before school started with the request that she develop instruction that dealt with language structures relative to what it means for a divisibility rule to work. In addition, we requested that the instruction be used in one class and suggested that this instruction be at the intuitive level in the context of division by 2, which would be familiar to students, so that the cognitive focus of students during the whole-class discourse could be on the language structures and not the numerical values.

Methodology

This paper is a comparison of data collected in one teacher's classroom over a two year period as she taught seventh-grade mathematics. The data collection during the second year was influenced by the analysis of the first year data providing a developmental perspective on the instruction and the discourse in this classroom. Our comparative analysis (Glaser, 1978; Glaser & Strauss, 1967) involved the coordination of observational data, the results of questions posed in written form to all students, and interviews (both teacher and student). The students were selected for the interviews based on their responses to the written questions presented to the class. All data sources were interactive since each point of data collection was influenced by the previous data collected.

In year 2 of the study, data were collected in two classes: 3rd period had explicit, although intuitive, language development relative to if-then structures at the beginning of the number theory unit; 4th period did not receive formal instruction on if-then structures. An overview of the key components of the relevant data collected over the two year period are summarized in Table 1.

Table 1: Classroom Activity Relative to the 2-rule

Year	Class	Key components of the data	Researcher Observations
1	6th period	Investigation of a student rule to test divisibility by eight. This rule became known as the 2-rule.	Socially aided construction of use and validity of 2-rule. Language difficulties revealed.
2	3rd period	Explicit introduction of structure of if-then including language.	The 2-rule presented by teacher; students were to validate.
2	4th period	Explicit introduction of language structures related to powers and factors.	The 2-rule presented by teacher; students were to validate.

All three classes responded to the same written questions designed to gather information on the students' understandings of the 2-rule for dividing by eight. These questions addressed various forms of the original 2-rule statement. Each of the questions had four choices (never, sometimes, always, or I don't know) and the students were to circle the answer that "best describes how you think". Three of the questions given to the students which are relevant to this discussion are:

1) If a number is divisible by 8, then you can divide it by 2, divide the answer by 2, and divide that answer by 2 again, getting a whole number answer each time you divide. If any of the answers is not a whole number, then the original number is not divisible by 8;
2) If a number has 3 factors of 2, then it is divisible by 8; and
3) If 2^3 is a factor of a number, then the number is divisible by 8.

Findings

In year one of the study, the construction of the 2-rule was moved out of the individual realm and into the social realm, thus the mathematical meaning was constituted in the social interaction (Voigt, 1989). This interactive perspective revealed the lack of understanding about the nature of mathematical rules as well as the formal mathematical language structures needed in the development of mathematical content knowledge.

The classroom discourse in year two of the study was characterized by the use of more formal mathematical language by the teacher and the students. In particular, the teacher used if-then statements more confidently than in year one of the study. In addition, if-then forms were used regularly by the teacher and students in the whole-class discourse as they explored numerical examples.

The student responses to the written questions were tabulated, simple percentages were calculated, and the results were recorded in Table 2. The sample sizes for 6th period, 3rd period, and 4th period were 25, 34, and 28, respectively.

Table 2: Percent of Correct Responses

Question	**Year 1** **6th Period**	**Year 2** **3rd Period**	**Year 2** **4th Period**
1	28	50	36
2	20	21	29
3	40	44	54

Data on Question 1, designed to evaluate the converse and contrapositive forms, indicates that a greater number of students in year two of the study (50% for the class receiving explicit instruction, 36% of the class not receiving explicit instruction) were able to successfully deal with the contrapositive form of the implication statement as compared to year one students (28%). One possible explanation for this result was that the development of this teacher's mathematical content relative to the 2-rule gained by the teacher's own reflection on the researcher's analysis. This development resulted in a change in the language the teacher used in the classroom discourse. That is, the classroom discourse was characterized by the usage of more formal mathematical language structures.

Evidence suggests that this change, together with the teacher's presentational style, may have affected the students willingness to participate in the construction of the content in a positive way. In addition, we believe that the evidence also suggests that the explicit format of the discourse in reference to the if-then statement in 3rd period also may have contributed to their performance (50%) on this portion of the survey as compared to 4th period (36%).

Further support for the conclusion that explicit mathematical discourse influences the content learned is shown by the results of Question's 2 and 3. Both of these questions were designed to evaluate the students familiarity with mathematical language and content, in particular factors and powers. Evidence indicates that both of the classes in year two of the study showed higher performance on these questions than the first year class. However, 4th period's performance (29%, 54%, respectively) was even higher than the 3rd period (21%, 44%, respectively). We believe that this result may have been due to the explicit instruction on factors and powers presented to this group.

Implications for Teaching

This study suggests that explicit attention needs to be given to the transition from regular, non-mathematical discourse to a more formal, mathematical discourse in the mathematics classroom. This explicit attention can and should address not only the mathematical content, but the nature of mathematics as well, through the discourse. Thus, curriculum development should be aimed at providing the teacher with background relative to the complexity of the language structures and the cognitive issues that are important to structurally support the content being taught.

Although this process is complex, we believe that it is attainable in today's classroom as illustrated by the teacher in this study. From year one to year two of the study she acquired some formal mathematical language needed to support mathematical discourse relative to if-then structures. Although the teacher received direction from the researchers, we do not believe that this change would have taken place without the teacher's own reflection on the content and the interactions in her classroom.

This reflection allowed time for this teacher to form connections among the content revealed through the discourse. The teacher's development allowed her to facilitate mathematical classroom discourse that supported the building of knowledge on the parts of all participants. We believe that this teacher changed her classroom because initially she valued the discourse as something beneficial to the student, she was willing to risk the implementation of the group construction of mathematical content, and she was willing to reflect on the relevant episodes that unfolded in the classroom. In time, the teacher came to value whole-class discourse as a means to evaluate student understandings.

We believe that the positive results witnessed in this classroom would not have taken place without support that gave the teacher specific information about her own classroom, the interactions in that class, and the complexities of the content she was teaching. These observations have implications for teacher inservice, suggesting that inservice provide support at an individual level for each teacher relative to that teacher's own classroom and personal experience.

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WHEN IS A STUDENT'S EXPLANATION SATISFACTORY?

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In a typical college mathematics course for, say, future engineers, the 'how' of calculations (which could in most cases just as well be carried out by a Computer Algebra System) is stressed far more than the 'why'. Recently, some college texts and teachers have begun to require students to write explanations for their computations. Apparently, this is done not only in the hope to improve students' understanding but also to improve their skills of communication: *Scientists and engineers need to be able to say precisely what they mean and why something is true*. (Lay: Linear Algebra, 1993)

Most college teachers know how to explain well; but much, if not all, of their knowledge about explaining is intuitive. If students are to give explanations, and if their explanations are to be assessed, the need for a meta-explanation arises: the need to characterize (good) explanations! The discussion group is proposed in order to tackle this and related tasks.

The following questions will be raised:

- What is an explanation? What is an acceptable explanation? What is a satisfactory one?
- What does the acceptability of an explanation depend on?
- Why do students (not) explain the mathematics they write?
- What do students consider as a satisfactory explanation?
- How do you explain to students what a (good) explanation is?
- How are explanations related to proofs?
- Whom, and of what does a student's explanation need to convince?
- How are student explanations related to understanding?
- Does giving explanations lead students to address meaning?

In order to provide some background, explanations from about 200 students enrolled in four calculus and linear algebra courses have been collected. Participants are invited to provide their own samples of student explanations.

It is expected that discussing the above and related questions will lead to the formulation of research questions concerning student explanations, and eventually to an analysis of student explanations, which includes a theoretical framework and criteria by means of which various components and aspects of student answers can be assessed.

HAVING TEACHERS ANALYZE CHILDREN'S MATHEMATICAL DISCOURSE TO REFLECT ON INSTRUCTIONAL PRACTICES

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Educators are being challenged to reconsider their conceptualization of mathematics and effective instructional practices. Through perturbation and reflection teachers can reconstruct the "process and profession of teaching mathematics" (Jakubowski, Wheatley, & Diaz, 1995). Teaching cases can be used as a vehicle for reflection and can perturb teachers to reconsider their own practice (Davenport & Sassi, 1995).

Consistent with the above, PERSPECTIVES, a professional development program for K-4 teachers, was designed to help participants become connoisseurs of mathematics education instructional practices (Eisner, 1976, 1991). Cases were used to give teachers opportunities to reflect on their beliefs and classroom practices. As a result, they described the influence that the mathematical experience and their posturing had on the mathematical constructions of their students.

This paper describes the impact of children's mathematical conversations (cases) on teaching practices. As these teachers began to focus on the children's mathematical conversations, they noted the differences between the students' mathematical constructions and their own assumptions about these constructions. Consequently, these teachers began to reflect on their beliefs about mathematics, mathematics teaching and learning, and to change their instructional approach.

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DISCOURSE AND THE DEVELOPMENT OF CONCEPTUAL UNDERSTANDING IN THE ELEMENTARY MATHEMATICS CLASSROOM

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The quality of discourse in the mathematics classroom is seen by many to be a critical factor in the development of conceptual understanding. The NCTM *Professional Standards for Teaching Mathematics* reflects the view of many mathematics educators: "The discourse of a classroom--the ways of representing, thinking, talking, agreeing and disagreeing--is central to what students learn about mathematics" (1991, p. 34). While there is an abundant literature advocating the virtues of discourse (cf. Ball, 1993; Cobb et al, 1992; Lampert, 1990), there are fewer studies that have specifically examined teachers' interpretations of these virtues and how these interpretations are manifested in the teachers' instructional practice. This study examined an elementary teacher's role in initiating and orchestrating the classroom discourse and the knowledge students acquired as a result of being participants in the community of practice established within the classroom.

The study was conducted during the spring semester at an elementary school in the Denver, Colorado area. The participants observed during each visit were the same group of twenty-four students and teacher in a multi-age elementary classroom (i.e., second and third grade students). The study utilized ethnographic research techniques for data collection (participant observations and interviews) and data analysis (taxonomic analyses of each data set, a componential analysis across data sets, and a realist analysis across data sets). The findings suggest that developing a community of learners, creating a discourse emphasizing student participation, and selecting rich mathematical tasks, are necessary, but not sufficient conditions for promoting the development of conceptual understanding in the elementary mathematics classroom. The teacher's interpretation of "quality" discourse differed in ways which had a significant influence on the understanding acquired by the students.

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THE REFLEXIVITY BETWEEN LINGUISTIC ACTIVITY AND COGNITIVE DEVELOPMENT OF CONCEPTS OF TEN: A CASE STUDY

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This study was intended to analyze concurrent changes in children's constructions of units of ten and their ways of talking and symbolizing. It took place in a second-grade class that was participating in a problem-centered approach to mathematics instruction. Small group work and teacher directed whole class discussions were the two forms of classroom organization.

The observation period consisted of fourteen lessons that took place in the course of six weeks. During this time, students solved money-related tasks with the use of coins. The whole-class discussions and the work of two pairs of children were video-taped. The selected children were interviewed prior to and at the end of the observation period.

The argument that linguistic activity and cognitive development interact reflexively is developed by re-interpreting within the specific context of classroom interactions, the concepts of languaging and consensual domain offered by Maturana (1978). The extensive analysis of Steffe, Richards, von Glasersfeld, and Cobb (1983), and Steffe, Cobb, and von Glasersfeld (1988) served to guide the researcher's inferences about children's constructions of ten and counting strategies.

During the observation period, children made significant cognitive advances in their construction of ten, developed increasingly sophisticated counting strategies, and made changes in their ways of talking and symbolizing. The data analysis supported the hypothesis that cognitive development and linguistic activity interact reflexively.

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EPISTEMOLOGY

ABDUCTIVE PROCESSES AND MATHEMATICS LEARNING

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This paper analyzes the zigzagging inferential processes (inductions, abductions, deductions) of a college student and a third grader as they solved tasks they found to be problematic from their perspective.

Introduction

This paper discusses the coexistence of abductions, inductions, and deductions that characterize mental human activity and especially mathematical activity. The researchers agree with Anderson's (1995) contention that the process of abduction is transitory and slippery, difficult to foster, impossible to teach, and probably easy to discourage. Such a contention echoes Mason's (1995) observation of the implicit meaning of the common mathematician's instruction "consider". Afterall, *consider* is a verbal expression that goes unchallenged by the listeners and is used by mathematicians because it requires neither inductive nor deductive justification. Rather, it expresses sudden, opaque, and evolutionary shifts of thought that support deductions and inductions. Since abductions are intermingled with inductions and deductions and never isolated cognitive exercises, "our trivialising curriculum turns everything into behaviour and avoids awareness, assumes deduction, tolerates induction and ignores abduction" (Mason, 1995, p. 4).

The philosopher and logician Charles Sanders Peirce (1839-1914) asserted that all mental processes are inferential and argued that induction, deduction, and abduction represent distinct stages of inquiry. Specifically, abduction furnishes the reasoner with a novel hypothesis to account for surprising facts; it is the initial proposal of a plausible hypothesis on probation to account for the facts, whereas deduction explicates hypotheses, deducing from them the necessary consequences, which may be tested inductively.

Since Peirce argued that abduction accounts for the engendering of new theories and conceptions (Fann, 1970), it appears that abductive processes play a prominent role in the thinking of students of mathematics of all ages and different degrees of experience. Indeed, the work of Polya (1945) is based on ideas consistent with the view that problem solvers reason hypothetically in the course of the solving process. Specifically, Polya identified heuristic reasoning as "reasoning not regarded as final and strict but as provisional and plausible only, whose purpose is to discover the solution of the present problem" (Polya, 1945, p. 113). Hence, the forming of an explanatory hypothesis requires creativity and supplies the learner with ideas from which to make conjectures about potential courses of action to carry out. The importance of such creative activity has been emphasized by Burton (1984), identifying the process of making conjectures as a component of mathematical thinking through which "a sense of any underlying pattern is explored" (Burton, 1984, p. 38). More recently, Mason (1995) has remarked on the importance of examining where learners' conjectures come from, suggesting that a fresh examination of the abduction process is warranted.

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[1] The order of authorship on this paper is alphabetical; the authors view this paper as a collaborative effort to which they both contributed in important ways.

Methodology

The two studies we draw from were conducted with different populations and incorporated contrasting research designs. The study of college students was conducted with nine students to observe the constructive paths that solvers follow during a problem solving activity. Each subject participated in a single interview, solving a set of similar algebra word problems while thinking aloud. In contrast, the study of third graders was part of a comprehensive one-year teaching experiment with six third graders who were interviewed 17 times each with results of a single interview reported here. The task posed to the students in this interview was to find the total number of pairs, triples, and quadruples from sets of different objects.

A. Marie's Inferential Processes: A College Student's Problem Solving Activity

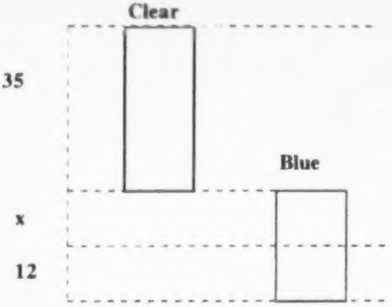
Subjects of this study were required to solve the following algebra word problem:

> **The surface of Clear Lake is 35 feet above the surface of Blue Lake. Clear Lake is twice as deep as Blue Lake. The bottom of Clear Lake is 12 feet above the bottom of Blue Lake. How deep are the two lakes?**

Subjects were then asked to solve eight follow-up tasks, each a variation of the lakes problem (e.g., similar but containing superfluous information, similar but containing insufficient information, similar but more general, etc.). These tasks provided opportunities for solvers to compare and contrast their actions across a range of similar problem solving situations, and involved their ongoing examination and re-examination of previously constructed strategies in the face of new problem situations. The generation of explanatory hypotheses by solvers was inferred as a means by which they framed and developed novel explorations about the efficacy of their solution strategies, with particular emphasis on ways their strategies needed to be revised and adapted to fit new situations.

Marie constructed a solution to the Task 1, using a geometric coordinate strategy to represent the unknowns (see Figure 1), an approach she successfully used while solving later tasks.

Figure 1: Marie's Solution to Task 1



$2(x+12) = x + 35$

$x=11$

Clear Lake = $x + 35 = 46$ ft.
Blue Lake = $x + 12 = 23$ ft.

After solving Task 1, Marie saw each successive task as similar to Task 1, anticipating that she would solve them much the same way as she solved Task 1. She experienced problematic situations whenever her anticipations of what to do proved faulty when actual activity was carried out. It was during these instances that Marie appeared to make abductions which had the impact of both changing the impetus of her solution activity and at the same time presenting novel opportunities for her to develop increased awareness about the efficacy of her previously constructed methods. For example, after solving Tasks 1 and 2 by using similar diagrams and solution methods, Marie attempted to solve Task 3, a problem with insufficient information, in the same way. However, she soon found herself faced with a problematic situation she had not anticipated:

> **Task 3: An oil storage drum is mounted on a stand. A water storage drum is mounted on a stand that is 8 feet taller than the oil drum stand. The water level is 15 feet above the oil level. What is the depth of the oil in the drum? Of the water?**

> **Marie:** I am going to draw a picture. Here is my oil stand ... water stand. And we have a water storage 8 feet taller. And here's level water. And here's the oil level. (Long reflection) So, solve it ... the same way. (She smiles, then displays a facial expression suggesting sudden puzzlement) Impossible!! It strikes me suddenly that there might not be enough information to solve this problem. (She re-reads and reflects on her work) I suspect I'm going to need to know the height of one of these things (Solver points to both containers in her diagram). I don't know though, so I am going to go over here all the way through.

Marie's anticipation that "the same way" would not work was followed by her hypothesis that the problem did not contain enough information, later refined to the hypothesis that she needed more information about the relative heights of the unknowns. By generating an hypothesis as a plausible reason for the unanticipated problem she faced, Marie adopted a new perspective in her activity that enabled her to consider aspects of the problem situation she had never before contemplated. While her hypothesis contained an element of uncertainty, it nevertheless helped her to organize and structure subsequent solution activity, whereupon she explored and tested its plausibility as an explanatory device. More importantly, these novel actions served as a learning source from which she eventually

developed more abstract criteria to evaluate her potential solution activity when solving later tasks.

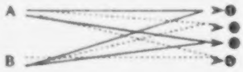
B. Michael's Inferential Processes: A third grader's Problem Solving Activity

Michael had participated in eight interviews on whole number and fractions before he participated in the interview reported here. This interview served as a warm up interview after the Christmas break. The task posed to Michael was used by Leslie P. Steffe in his teaching experiments and was used with his permission. The child was presented with two, three, and four sets of cards (letter-, number-, figure-, and color-cards) in this order. The cards were displayed over different pieces of construction paper as a means to define a boundary for each set of cards. The number of cards on each set was changed to vary the degrees of difficulty of the task.

Michael was asked in chronological order for the number of pairs, triples, and quadruples that might be possible to make if he were to choose one card from each set. When he was first asked what a pair was he said "it's two of a kind" and, given his belief on the need of some kind of resemblance among the elements, he resisted the possibility of making pairs. Once an agreement was reached about choosing one element from each set and only after some trials, he suddenly found a strategy that consisted of pairing each card of the set of letter-cards and matching it with all the number-cards.

When solving a particular task, Michael silently moved his eyes and head from left to right; when explaining, he moved his fingers over the cards while repeating the pairs. This gestural action seemed to reproduce some type of mental image of his strategy to combine the elements of the sets given the absence of pencil and paper to keep his records. Without this mental image, it would have been impossible to keep the record straight while in the process of verbalizing and gesturing. For example, when the set of letter-cards was A, B and the set of number-cards was 1, 2, 3, 4 his answer was:

Michael:



[the arrows in the diagram represent the sweeping action of his right hand]
A1, B1; A2, B2; A3, B3; A4, B4 [keeping track of the counting in his fingers]

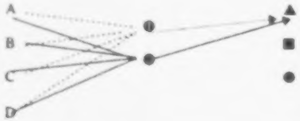
Up to this particular point in the interview Michael had proceeded in an inductive manner, supported by his first "organizing" abduction (coupling each letter-card with all the given numbers). He was then giving the letters A, B, C and the numbers 1, 2, 3, and 4. He looked at the cards and said, "This is hard". He thinks for a second and with a spark in his eyes he says, "Oh! 12; 3, 6, 9, 12." The researcher ask him to explain.

> **Michael**: Because there are three letters here [showing the letter-cards A, B, and C]. A, B, C can be paired with 1 and that is three; A, B, C can be paired with 2 and that is three; A, B, C can be paired with 3 and that is three; A, B, C can be paired with 2 and that is three. [Showing each of the number-cards he says] 3, 6, 9, 12.

Michael quickly saw a way of curtailing the counting of the pairs without actually making them. The novelty of structuring his counting by this means was a "structuring abduction" because it allowed him to perceive this task in a structured manner. This abduction was preceded by a less sophisticated chain of abduction-induction that served as a basis for a more complex interpretation of a less simple task. The question is if this chain would foster more complex chains abductions, inductions and deductions to allow Michael to find the number of triples and quadruples when given three and four sets of cards, respectively.

In the following episode, Michael was given the set of letter-cards A, B, C, and D; the set of number-cards 1, and 2; and the set of figure-cards ▲,■,● and he was asked to find the number of triples that would be possible to make. He took some seconds, move his head from left to right at the unison with his right hand and said "twenty", then he went on to explain (as before the arrows in the diagram below indicate Michael's gesturing.)

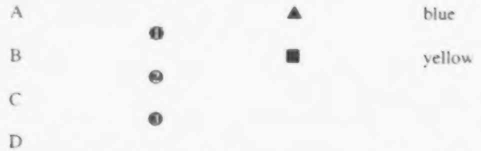
Michael:



> All four can go to 1 and to triangle, that is four; all four can go to 2 and to triangle that is eight. [Then pointing out at each of the figures from top to bottom he counts] eight, eight, and eight is twenty-four.

The task posed to Michael prior to this one was to find the number of the triples as the only figure was the triangle. His answer eight was followed by his explanation "because the number of triples is the same as the number of couples if you take this [he takes the triangle in his hands] away." The above solution seems to have been grounded on the prior solution that was in itself accounting for a more sophisticated structuring abduction. His above explanation indicates another abduction-induction chain in Michael's reasoning. Would Michael be able to carry out his inductive reasoning grounded in his transitory and context-bounded abductive processes if asked to make quadruples with four different set of cards?

Teacher: How many quadruples can you make here?



Michael: [Look at the cards] This is hard! [After some seconds] Let me see how many couples are over here [takes the triangle and the square in his hands] twelve. Twelve and twelve is twenty-four triples. Then twenty-four and twenty-four is forty-eight quadruples.

Michael's solution clearly indicated a deductive action intimately bounded to his prior chains of abductions and inductions. This deduction was determined by his own constructed knowledge of making triples out of couples that he immediately carried to the higher level of making quadruples out of triples prior to making them.

In summary, Michael's solutions to the different but complementary tasks illustrate abductions as a kind of theory-forming or tentative grasping of hypothesis that occur synergistically with inductions and deductions and not as isolated cognitive exercises.

Conclusions

The analyses presented here focus on the role of abduction as a mediating and transformational influence in the inferential thinking of the students as they faced mathematical tasks that involved relational or totally new problem solving situations. Other mathematics educators have analyzed the conceptual "novelty" idiosyncratically introduced by students while solving problems (Burton, 1984; Mason, 1995; Steffe, 1991; Tall, 1991). The analyses reported here help to extend results of those earlier studies by documenting more specifically the ways that abduction facilitates the evolution of the student's conceptual knowledge.

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TEACHING MATHEMATICS TO *ALL* STUDENTS: THE MATHEMATICAL EXPERIENCE AND LEARNING OF A THIRD GRADER

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This is a preliminary report of one of the six case studies conducted in a third grade classroom. Through the story of a student who was considered behind her peers in her mathematics concept development, I hope to highlight the challenges of teaching mathematics to *all* students. Finally, I present some questions with which I am still struggling.

At the heart of the current mathematics education reform effort is the commitment to provide ALL students with the opportunity to experience the value and power of mathematics, and to become confident in their abilities to use mathematics (National Council of Teachers of Mathematics [NCTM], 1989). The emphasis on problem solving, communication, connection and reasoning calls for a mathematics program which is different from the traditional lecture/seat work type of instruction.

While there have been studies on the challenges teachers face when attempting to change their mathematics teaching practices, studies on individual student's experiences and learning in such a setting are not yet widely available. In particular, few researchers have listened to students who have not been successful with school mathematics. Reflecting on a one-year study of a mathematics program which exemplified many recommendations from NCTM, Parker (1993) noted that while many students in the program excelled beyond expectation, it was difficult to determine whether the program met the needs of the low achievers. A better understanding of children's experiences is essential in the continuing development of quality mathematics programs.

The Project

This project was conducted at a downtown small city school with mixed student population in terms of ethnicity and SES. It was a collaborative effort among three members: an experienced third grade teacher, a Support Teacher, and me. The team worked on the following recursive tasks: 1) analyzing student experience and learning, 2) adapting curricular materials and instructional activities to better meet student needs, 3) developing strategies to further the investigation of children's experience and learning both in the interview sessions and as part of the classroom instruction.

The theoretical framework of this study was grounded in the constructivist view of knowledge and learning (von Glasersfeld, 1987), and symbolic interactionist view of social interaction (Blumer, 1969). This study explored the following related issues in the classroom:

1. How did students at different levels of mathematics sophistication participate in mathematics instruction? What learning opportunities existed for each student?

2. How did these students perceive their experience of learning mathematics? In particular, we focused on their concepts of self (e.g., self-judgment and self-control) and role-taking (e.g. interpreter, elaborater).
3. What advancements did these students made on their arithmetic concepts, and ability to solve problems and communicate mathematics ideas in light of their participation and experience?

To investigate these questions, I conducted case studies of six students throughout the 1995-1996 school year. The primary research techniques included participant observation and structured and unstructured interviews. I was in the classroom daily for at least half of each day, for the whole day when necessary. The data was gathered through video recording of daily math lessons, interviews, and a comprehensive portfolio of student written work. The target students were selected based on an interview assessment of mental computation, the Stanford Achievement Test (SAT) score, and on observations made by the research team. There was an attempt to select students so that they would reflect the diversity of the school population in terms of ethnicity, SES and native languages. This paper will present some preliminary analyses on one student, Sophia.

The setting:

The school's population, comprised of neighborhood, open-enrolled, and ESL students, reflects the diversity of the city in New York State. This particular third grade classroom had 20 students, 9 girls and 11 boys. There was a good mixture in terms of ethnicity (10 minority) and ability (1/3 scored in each above 85, around 50, and below 15 percentile on the SAT), as well as SES status (60% receive free lunch). In addition, there were three ESL students and two students diagnosed learning disabled, one with a speech and language disability, and one with a serious emotional disturbance. As an inclusion school, no students were pulled out for separate instruction, rather, a half-time special education teacher was brought in to assist with the instruction. With an additional special education aid, there were three adults at almost all the math periods.

The third grade teacher has worked in the district for 19 years. With an undergraduate degree in math, he is also very knowledgeable in the area of science and technology. The math curriculum was drawn from various sources depending on students' needs and district guidelines. For example, we used multiplication and division unit from *Investigations in Number, Data, and Space* (Tierney, Berle-Carman & Akers, 1995), geometry units from *Marilyn Burns Replacement Unit* (Rectanus, 1994) and several number units from *Purdue Problem Centered Mathematics Programs*, as well as some teacher-made materials . We also used various games, like NIM, box game, and various card, dice and chip games from Kamii's book (Kamii, 1989), etc.

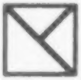
Sophia's initial arithmetic concepts

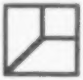
The first interview with Sophia on 9/14/95 revealed her limited understanding of numbers and operations. Sophia was not familiar with simple addition facts, like 2+3, or 3+7. She used a counting-up-by-one strategy to solve "6+8" correctly but was unable to use the same strategy for "13+13" because she ran out of fingers. When using base 10 blocks to solve 25+25, she could correctly lay out two groups of 25 cubes in the form of two sticks, five cubes, two sticks, five cubes. But then she counted, "10, 20, (pointed to the two sticks), 21, 22, 23, 24, 25, (pointed to the five cubes) 26, 27 (pointed to the two sticks), 28, 29, 30, 31, 32, (point to the five cubes).

For simple subtraction problems like 10-5, 7-4. Sophia used her fingers. She recognized the 5-finger pattern, but needed to count twice to make sure she had three fingers remaining. I asked Sophia to put up 8 fingers on top of her head. It took her a while to put up 5 fingers on one hand and 3 on another hand. Then I asked her "What is 8 take away 5?" she put down five fingers one by one, then said 2.

Sophia's participation in mathematics instruction

Sophia was on more equal footing with her peers when the focus of mathematical investigation was not arithmetic (for example, geometry, number patterns, logical reasoning) or the numbers involved were manageable with manipulatives. In those instances, Sophia liked to participate in class discussions. For example, she was eager to tell the class that the shape (see below) on the right looked like "a kite going down," while the shape on the left reminded her of "a slide" during a Quick Draw activity. She was also happy to share with the class the pattern she noticed when coloring the multiples of 3 on the hundred chart.





However, her concrete-based arithmetic concepts limited her participation in class discussion on mental computation and thinking strategies which was an important part of the mathematics program. For example, at the beginning of the semester, one common opening activity was for students to guess how many cubes were in each of the teacher's hand if there were 10 cubes altogether. One day, after four students had offered guesses like "5&5, 8&2, 7&3," Sophia was asked to make a guess. She remained silent for a long time and the teacher had to keep other students quiet by reminding them "not to cheat Sophia of her thinking." She finally said there was 7 in the right hand. "So how many would be in the other hand?" the teacher asked. Sophia counted on her fingers slowly and said, "4." The other students disagreed and pointed out that 7 and 4 would make 11. With little facial expression, Sophia agreed with the correction. This instance showed that Sophia was still struggling with her counting strategy for solving missing-addend task.

Sophia's experience as a math learner

Sophia had little confidence in her math ability. She told me during our first interview that she had a short memory so she could not do math well. Even after working with her for months, she would still hesitate to offer her thought.

Despite her lack of confidence, she was quite happy during math time if she was given tasks which were within her reach and was allowed to move at her own pace. The latter turned out to be the most critical factor. She became very upset when partners rushed her to make a move, and similarly, when she was given only 15 minutes to do the practice test. From Sophia's perspective, even though she might not know how to do the problems, it was *wrong* not to give her enough time to finish. To get an idea of Sophia's need for extra time: all her classmates finished 65 problems on the state-mandated test by the end of three hours, but Sophia was still working on #32.

Sophia's development of arithmetic concepts

First of all, Sophia became a lot more confident in using manipulatives. When given the unifix cubes in ones and tens, she could make the number 26 by getting out two tens and six ones easily. When solving "26+26," she would sort the tens and ones into different piles, then counted, "10, 20, 30, 40, 41,...49, 50, 51, 52." When doing a subtraction problem like "14-8," she no longer pulled out 14 ones and 8 ones with no idea what to do next like she did at the beginning of the school year. She counted out 14 ones, took away 8, then counted what was left. Second, she figured out some number patterns, such as 10+7=17, 30+4=34, and remembered some number facts like doubles. Third, she developed the intention to find an easier way to solve problems. For example, when solving "17-9," she claimed that she wanted to find an easier way to solve the problem. Although her easier way turned out to be putting all the 17 cubes together in one long stick, which did not seem to be easier to us, her intention to do so was worth noting.

Meeting the learning needs of Sophia

In this session, I will first describe several important features of a mathematics program which can meet Sophia's needs as an individual learner based on the results of my preliminary analysis. Then I will talk about the challenges of providing such a program as I reflect on the one we provided during the 1995-1996 school year. Finally, I will present some questions with which I am still struggling.

The first important feature of the "ideal" math program for Sophia is to have plenty opportunity for her to develop her arithmetic concepts through concrete experience. Second, there needs to be a good mixture of topics from non-arithmetic areas such as geometry, measurement, and patterning which will give Sophia more opportunity to actively participate in mathematics class discussion. Third, Sophia will benefit from a partner who is at about the same level in mathematics development, and is patient and willing to articulate his/her thoughts. Fourth, the learning environment needs to be established such that speed is less important than understanding.

The first challenge we were facing was the range of student ability in their arithmetic concepts. The mental computation strategy interview we did at the beginning of the school year indicated that, we had 13 students who could solve two-digit addition problems effectively and were at various stages in developing similar strategies for subtraction. But the remaining 7 students were at about the same level as Sophia whose interview performance can be summarized as the following:

* The only strategy used to solve addition and subtraction problems without manipulatives was to count up or down by ones.
* Easily lost track of counting when the addend was larger than 20.
* Easily lost track when the second number was larger than 10.
* Did not know simple facts like 3+5, 10+2.

Looking back, even though we spent about 1/4 of our instructional time directly focusing on tasks which supported the latter group in developing their arithmetic concepts, we were not satisfied with their progress. On the one hand, we were pleased that we did offer students a much more balanced, varied, and challenging program as NCTM suggested; on the other hand, we wonder if more could be done to provide consistent support for the development of arithmetic concepts even though the main topics were geometry or data analysis.

The second challenge to meet Sophia's needs was to find her a good partner. During the school year, we allowed the students to choose their partners most of the time. Out of Sophia's own choice, she most often worked with Kelly, the best math student in the class, or Kim, a girl who was just a little more advanced in math than Sophia. When Sophia and Kelly worked together, Kelly was acting like a tutor for Sophia. She explained things and posed questions for Sophia to think about. But as a good friend, Kelly often ended up showing Sophia how to do the problems. When Sophia and Kim worked together, frequently, neither of them knew how to get started or could sustain their effort on their own for more than 10 minutes without a teacher's assistance. Further analysis is needed to look at whether either type of interaction was productive. But we were not totally pleased with Sophia's engegement in either case.

The third challenge to meet Sophia's needs was to establish a high, yet realistic expectation for Sophia's progress. Should we have been satisfied if Sophia solved only one word problem, "18 kids will make 3 teams. How many kids will be in each team?" in 20 minutes while most her classmates could finished 4 or more similar problems with much harder numbers in the same period of time? Should we have provided more opportunity for Sophia to focus on numbers and operations directly rather than trying to present everything in a word problem or a game format, knowing that making sense of the additional information might be difficult for Sophia? Knowing Sophia, we were certain that she would not be happy in a pull-out program where she would be treated differently. We were also certain that she would not like math as much if she were in a more traditional program where computation was emphasized. But we were less certain whether she could be taught to perform more on computation tasks in either of those cases. Instead of keeping up with her classmates, in many ways, the gap in mathematical knowledge widened between Sophia and most of her peers. The challenges for a fourth grade teacher to meet Sophia's needs will be even greater. Finally, what about the ongoing increase in challenge for Sophia? How can we keep her succeeding and learning if she continually faces bigger challenges?

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ALL-AT-ONCE UNDERSTANDING AND INTER-ACTION IN MATHEMATICS*

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In this paper, the interactions among a parent, a child, and their environment are used to focus our discussion of mathematical cognition as an all-at-once phenomenon. Enactivist inquiry as used in our research suggests that cognition is a complex phenomenon arising from interactions among persons in an environment. Further we suggest that mathematical cognition is triply-embodied: within the person as he or she uses, reorganizes and modifies his or her own structures to bring forth a mathematical world; through a person's interactions in that world; and through a person's interactions with the mathematical knowing of historical and contemporary communities. Our research points to the value of viewing mathematical understanding as embodied actions and interactions among persons within a local setting.

Krista, a 10 year old and her father, Dave, had been participants in an extracurricular mathematics program for parents and their children for over a year. Once a week they met with other parent/child pairs and a leader to engage in mathematical activity. On this occasion, the parent/child pairs were given a set of dominoes and worked from the following prompt:

Find the number of different ways a set of 2x1 tiles can build a path that is two units wide.

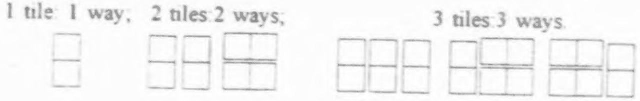

Figure 1

After illustrations of possible paths for sets of 1, 2, and 3 tiles were drawn on the board (Figure 1), Dave asked Krista, "What do you want me to do?" Krista responded, "As we do it, we should probably look for a pattern and that might help us with the other ones" (**I**).

These opening remarks foreshadow the nature of their mathematical activities and relationship for the remainder of the session. Dave and Krista's actions and interactions while investigating the above prompt are used in this paper to focus our discussion on mathematical cognition as an all-at-once phenomenon.

Several features of their interaction, including their physical activities and record keeping (see Figure 3), became prominent as Dave and Krista re-examined the sets of 0, 1, 2, and 3 tiles on their own. As they continued on to examine a set of 4 tiles, these and additional features were revealed. Krista began this portion of the investigation by making an unarticulated conjecture. "

If I'm right about this, then, if 4 follows my theory– Okay, let's see" (**II**). Krista then makes their first arrangement with 4 tiles and Dave records it. They continue to make and record different arrangements with 4 dominoes until Krista says, "I don't think there's any others."

Dave asks, "So how many do we have?"

Together they count "1, 2, 3, 4, 5."

"Shoot! That doesn't follow my theory," says Krista (**IV**).

"No. So far it just blew it out of the water."

Pointing to the chart (Figure 3) Krista explains her conjecture, "See I thought 1-1, 2-2, 3-3, 4-5." She shakes her head when the pattern breaks down at 4-5.

"It looked pretty close. I guess we can't stop yet, but we might find a pattern yet. Okay–one with five" (**V**).

Dave and Krista persist in using pattern making, record keeping and, as seen in the following segment, informal mathematical language to generate mathematical ideas to investigate a set of 5 tiles.

D: Okay, so what do you have?
K: 1, 1, 1 –
D: 1, 1, 1 –
K: 2 sideways.
VI D: Blip blip – Okay, what do you call the one sideways or something else?
K: Blip blip.
D: Okay, the blip blips.
K: Blip blip, 1, 1, 1.
D: Blip blip, 1, 1, 1.
K: Comma.
D: Comma. Thanks. That's a good recording technique. What are we doing now?

They find 3 other arrangements for 5 tiles and then Dave notices one of Krista's strategies for making arrangements.

D: Oh, yeah. That's a good pattern Krista because usually you're always kind of doing the mirror image of them. You figure out one and then do the mirror image. 1, blip blip, blip blip. Cause your other one was blip blip, blip blip, 1.
VII K: Do we have blip blip, 1, blip blip?
D: Blip blip ... No, sir. Good job. Kay so it's blip blip, 1, blip blip? Okay.
K: I think that's about it. What do you think?
D: Um, I can't think of any new ones. Well, we could always come back to it later on if we don't see a pattern – right?
K: Yah.

Figure 2

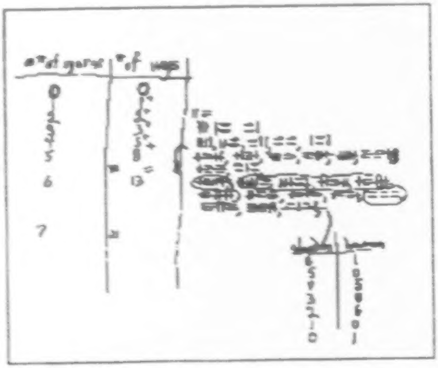

Figure 3

On Understanding

It is clear from the "data" above that Krista and Dave were engaged in mathematical activity and doing so with growing understanding. Interpreting the data using the Dynamical Theory for the Growth of Mathematical Understanding (Pirie and Kieren, 1994) allows us to consider both the structural and functional aspects of understanding (Hiebert et al., 1996) so that we may trace both the changing thinking and acting structures of Krista and Dave, and look at elements in their interaction which prompt changes in understanding.

In their mathematical activity, Krista and Dave deliberately use tables and specific recording devices to *make* or *form images* of the mathematics on which they are working (e.g., Krista's opening statement (I), and Figure 3). At II, Krista gives evidence that she has *an image* of how this mathematics works which she later proves false (IV). This incident suggests that one's images and understandings do not always prove to be correct.

Krista and Dave's *folding back* to change or extend their understanding (Pirie and Kieren, 1994) is in evidence here as well. We note at V that Dave *invokes* new image making and keeps them going in the face of Krista's disappointment. Record keeping (Figure 3) and the informal language development (Figure 2, VI) fosters such image making. As seen at VII and in the problem extension in the lower right hand corner of Figure 3, such actions allow for making and testing new images.

Concepts from the Dynamical Theory allow us to observe Dave and Krista's changing understanding as a non-linear pathway of making, having and elaborating images which they each construct and use. But this is not the whole picture of Dave and Krista's growing understanding. Such a mentalist view allows the effects of the environment to fade into the background by ignoring the roles of the physical and verbal interaction between Dave and Krista, and between them and the setting which included, for example, the prompt, the dominoes, and the artifacts of their work. To more fully comprehend their embodied mathematical understanding we can observe and interpret it in light of such interactive features. An enactive view of cognition is useful for such a task.

On Enactivism

To account for the complexity of cognition, we are trying to embed our interpretation of mathematical understanding and activity (here a parent/child mathematical activity) in a broader *enactivist* inquiry. Enactivism is underpinned by the premise that cognition is not the representation of a pre-given world by a person, nor is it simply a phenomenon emerging out of an individual's brain or body. Rather, cognition occurs in the interaction between persons and their environment and, at the same time, it is fully determined by the person's ever changing biological, social and phenomenological structure (Maturana & Varela, 1987; Varela, Thompson and Rosch, 1991). Further, we suggest that mathematical cognition is triply-embodied. First, *in-person embodiment* involves a person using, reorganizing, and modifying his or her own structures and schemes to bring forth and act in a mathematical world. Second, a person's body necessarily places him or her in a world (*person-in embodiment*). Thus, the inter-actional dynamics of a person in a world are fully implicated in mathematical cognition. Finally, a person is *embodied in the body of mathematics*. That is, learners interact with the mathematical knowing of their community, which includes other students, the teacher, and curriculum developers, as well as the historical and contemporary community of persons engaged in the practice of mathematics. Thus, a person's mathematical cognition is *coemergent* with the environment in which it occurs. A person's social interactions and the environment in no way prescribe cognition, but nevertheless act to occasion it.

Enactivist inquiry suggests that we do not investigate cognition as mathematical understanding in isolation, but rather we examine understanding, the setting and the inter-action *all at once*. Consider how the prompt can be observed to provide occasions for understanding actions to occur. The tiling prompt Dave and Krista worked with in this particular session is an example of a "variable entry prompt" (Simmt, 1996). Notice how Krista's action of manipulating tiles to find patterns, Dave's recording of the patterns, and their noting of the number of arrangements for a set of tiles in a table format were all viable actions in light of the original prompt (an environmental constraint). Notice also how in acting with the prompt and each other, Dave and Krista formulated and reformulated – what an observer might view as – "problems": looking to see if the tilings matched the conjecture; finding a function that would predict the number of arrangements for *n* tiles; developing a reporting and recording strategy; and exploring why the tilings grew in the way they did.

The formulation of various problems arise out of constraints from the prompt but also out of the participants' discourse. In this example, Dave and Krista engage in a mathematical conversation between parent and child (see Gordon Calvert, 1996). Dave and Krista's mathematical activity was not about finding a "right answer"; instead, their conversation can be seen as a process of coming to a deeper understanding about issues of mutual concern. These issues, or "problems", as noted above, form the topic(s) of their conversation. In this session, Dave and Krista are observed to be investigating a function occasioned by the prompt. The complementary roles they form (Krista manipulates the dominoes while Dave keeps records), and the language they invent (Figure 2) allow them to maintain their relationship with one another through efficient actions and communication. Their mutual concern for the topic, the similar image they have formed of the problem, and their collaborative relationship allow them to jointly focus on the topic of conversation – the emerging number pattern. As Dave and Krista maintain their relationship with one another and with the topic of conversation, a recursive loop is formed as their actions and utterances in the past serve to occasion and support their further activity and frame their mathematical understanding. Thus, their continuing conversation serves to shape the world of mathematics they bring forth.

Concluding Remarks

It is common to think of a person's mathematical understanding – say understanding of Fibonacci sequences – as an acquisition. Or, even if one takes a problem solving approach, it is easy to think about mathematical activity as occurring in brief responsive bursts. The "data" presented here, the dynamical understanding interpretation of that data, the enactive view of cognition, the participants' discourse, and the setting in which that cognition is embedded all ask us to think in a more complex way about mathematical problem solving and mathematics learning in action. As observed in many mathematical settings with different students (e.g., Pirie and Kieren, 1994), Krista and her father, Dave, were not simply solving a discrete mathematical problem, they were bringing forth a world of mathematical significance with one another in a sphere of behavioural possibilities. Their mathematical understanding and actions were occasioned by a prompt and by materials which they interpreted and re-interpreted into the problem(s) that formed the topic of their mathematical conversation.

What implications does such a view and interpretation have for mathematics education research and teaching? It reminds us that mathematical knowing actions and understanding are part of on-going human activity. In fact, the situation of that activity, the interactions which take place, the nature of the conversations and the reasoning patterns are all fully implicated in the mathematical understanding. It also promotes the observation of mathematical understanding as embodied action arising in a setting and in interaction. Thus, it suggests how mathematical activity, understanding, the setting and interaction may be thought of all-at-once.

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VYGOTSKIIAN PERSPECTIVES ILLUMINATING A MATHEMATICS TUTORING PROGRAM FOR INNER-CITY CHILDREN

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Background of Study

This study reports a preliminary model of teaching/learning within 1-to-1 learning situations. The model was developed to describe results of an After-School Mathematics Tutoring Program (ASMT Program) using a Vygotskiian perspective that emphases the importance of cognitive, affective, and motivational supports for learning by tutors and tutees. The six-month study involved ten under-achieving students (tutees) from a inner-city school in a low social-economic area of a large city in the midwest. Students from grades kindergarten through third grade were tutored by teachers aides and upper-grade students (cross-age tutors) for an hour after school two days a week.

Effective Teaching/Learning Tool

In order to express the relationship between learning and development, Vygotsky developed the concept of the zone of proximal development (zpd). The zpd is usually analyzed with respect to cognitive functioning and omits much consideration of the life histories of the individual that influence the learning situation. We have found that needing to motivate tutors and tutees is a major issue early in the tutoring program.

Motivation is heavily related to each individual's cultural/historical background (e.g., as manifested in the economic and social status and other aspects of the actual living situation). Low self-esteem and bad role models within their daily lives can lead children along paths of their lesser selves. We see many daily examples of inner-city children who fall victim to such circumstances.

The ASMT program director cast the tutoring environment as a large family in her language and affective stance toward the tutors. She took on the role as a parent who cared enough to point out negative behavior and its effect on the younger children who looked to them as role models. Tutors were cast in the role of responsible, caring, and competent older siblings. Building environments in which inner-city children can be told it matters that they be a good role model can have immediate results that enable them to take on more active roles in developing the cognitive functioning and self-esteem of younger students. Building environments in which such students get support to be a good model is an important ingredient that is often overlooked as an effective teaching/learning tool.

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A MODEL OF THE CONSTRUCTION AND PRAXIS OF THE EPISTEMOLOGICAL MATHEMATICAL SYSTEM OF A MATHEMATICS CHILD PRODIGY

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This research is a case study of a mathematics child prodigy. Its aim is to answer three questions: "How does the participant child prodigy explain mathematics and mathematical thought?", "How does the participant child prodigy think about mathematics and learn mathematics?", and "Is there sufficient evidence to corroborate the proposed epistemological model (or a modification of it)?" The research involved four years of mathematical contact with a mathematics child prodigy and included videotaped mathematics mentoring sessions with the participant; audiotaped interviews with parents, teachers, and professors; and mathematical writing, journal entries, and other data involved in some way.

The study has found that the best explanations of the participant's mathematics and mathematical thought came from his written solutions and proofs and from his oral explanations and answers to questioning. It was observed that he was better able to explain his mathematics and mathematical thought when he was asked for explanations or verifications than he was when asked for a "proof," which seemed intimidatingly formal.

A proposed model of mathematical knowledge acquisition and use is at the core of the research and has evolved throughout the study. The final form of this learning theory model is described here. The child prodigy participant constructed his own mathematical systems consisting of knowledge graphs (information processing knowledge representations based upon graph theory and artificial intelligence state spaces). He then made use of these knowledge graphs through problem-solving search strategies similar to those employed with artificial intelligence state spaces (e.g., depth-first search).

Three sets of factors were seen to affect both the construction and use of the knowledge graphs. Motivation, energy, and focus provided the impetus for the participant's mathematics. Organization and memory facilitated the structure of the mathematical system. Lack of formal mathematics training contributed to his creativity and the nonstandard ways in which he constructed and used mathematics.

FUNCTIONS AND GRAPHS

UNDERSTANDING OF FUNCTION NOTATION BY COLLEGE STUDENTS IN A REFORM DEVELOPMENTAL ALGEBRA CURRICULUM

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This research explores the impact of using the function concept as a core idea in developmental mathematics. The initial phases of the study profile the growth in developmental algebra college students' concept image of function. Students' use and misuse of function notation is the main topic of this paper. Pre- and post-course questionnaires and task-based interviews are used to profile student understanding. Results indicate the ability of the target students to acquire some depth in their recognition of function notation, though the fragility of their concept image is consistently evident.

Introduction

A vexing problem facing U.S. college mathematics departments today is the sizable percentage of the student body that must begin their college career in a non-credit mathematics course. Courses that fall under this umbrella include arithmetic, geometry, beginning algebra, and intermediate algebra. In 1990, 58% of the 1.3 million community college students taking mathematics were enrolled in Intermediate Algebra or below (Watkins *et al.*, 1993). One frustrating aspect of this is that many students in these developmental courses have been exposed to the material in high school courses or in courses at other colleges. What can be done with college developmental mathematics courses to rescue these students and provide them with the understanding necessary to succeed at credit level mathematics courses?

This paper focuses on the results of a study on developmental algebra students who have been exposed to a "reform" developmental algebra curriculum in which "function" is the focal concept. While the purpose of the entire study is to develop profiles of student understanding of function along both breadth and depth dimensions, this paper discusses the depth to which such students acquire an understanding of the notational aspect of the function concept.

Theoretical Framework

The function concept has been a major focus of attention for the mathematics education research community over the past decade. (See Dubinsky & Harel, 1992, for example.) Schwingendorf *et al.* (1992) contrast the *vertical* development of the concept in which the process aspect is encapsulated as a function concept and the *horizontal* development relating different representations. DeMarois & Tall (in press) refer to these as *depth* and *breadth* respectively (noting that increasing depth implies higher levels of cognitive abstraction) and investigate the way in which a student's concept image of function can be described in terms of these two dimensions. The depth dimension has been discussed extensively, including Dubinsky's Action-Process-Object construction, Sfard's (1992) interiorized process which becomes first condensed and then reified as an "object-like entity," and Gray & Tall's (1994) procept theory. The breadth dimension refers to the various ways of thinking about a mathematical entity. These include not only geometric, symbolic, and numeric aspects, but also spoken, written, enactive, informal, and notational aspects. Profiles of students are developed by analyzing their responses in terms of both of these dimensions. We discuss the notational aspect only in this paper.

Instructional Treatment

Students involved in the study were enrolled in "pilot" sections of beginning algebra at 4 different community colleges. The key "reform" component of the pilot sections was the text material. The text (DeMarois, McGowen & Whitkanack, 1996) focuses on student investigation of problems based on a pedagogical approach that uses a constructivist theoretical perspective of how mathematics is learned (Davis *et al.*, 1990). The text's authors subscribe to the theoretical perspective that the main concern in mathematics should be "with the students' construction of schemas for understanding concepts. Instruction should be dedicated to inducing students to make these constructions and helping them along in the process." (Dubinsky, 1991, p. 119) Each unit begins with an investigation of a problem situation. Following the gathering of data, students work collaboratively on tasks based on the investigation activities. A discussion in the text summarizes essential mathematical ideas. The instructor orchestrates inter-group and class discussions of the investigations. Explorations are assigned to broaden and deepen the knowledge students are expected to have constructed during successive steps of the cycle.

The materials focus on development of mathematical ideas using a core concept of function. Function is initially defined as "a process that receives input and returns a unique value for output." (DeMarois, McGowen & Whitkanack, p. 92) Each function is based in a problem situation. Functions are investigated numerically, graphically, and with function machines before the symbolic form is created. Tables, equations, graphs, function machines, verbal and written descriptions are all used to analyze relationships.

The Study

Students at four community colleges completed written function surveys at the beginning and the end of a "reform" beginning algebra course during the spring semester of 1996. Subsequently, several students at each site participated in task-based interviews. Both the written surveys and interviews were used to profile students' concept image of function. We discuss some preliminary findings concerning the notational aspect along with the implications for curricular reform at the developmental mathematics level.

Analysis

One hundred forty-nine beginning algebra students completed a pre-course function survey. No questions on function notation were included since it was assumed that most students surveyed had not encountered the function concept previously. Data gathered from the question "Explain in a sentence or so what you think a function is. If you can give a definition for a function, then do so." verified this assumption. Using the categories as described by Breidenbach *et al.*, 95% of the respondents either left the question blank (72) or fell into the prefunction category (70). The post-course survey was completed by 82 students. Several questions focused on student interpretation of function notation. Interviews provided follow-up data. This paper discusses data from two such interviews: with DB and with LAF.

On the post-course written survey, students responded "true" or "false" to the following statements: "Suppose that f is the name of a function and x is the input to that function.

a. $f(x)$ represents the output of the function when x is input.

b. $f(x)$ represents the product of f and x.

c. $f(x)$ represents the rule you follow to find the output."

Of the 82 respondents, 73% answered "true" to part a, 68% answered "false" to part b, and 51% answered "true" to part c. Twenty-six percent responded "true" to a, "false" to b, and "true" to c. Both DB and LAF responded "true" to part a and "false" to parts b and c. Thus, a majority of the students were able to interpret $f(x)$ as the output of a function and not as notation for a product. About half connected the notation with a rule to be followed. The two interviewees did not see $f(x)$ as representing a rule to be followed. Let's look at the responses of DB and LAF to a similar question during the interview. The interviewer asked "What do you think when you see the notation $y(x)$?"

> DB: Well when I first saw it I thought y times x. Now it is easier to figure out that it is y is dependent on the value of x.
>
> LAF: y is dependent on x.

Both define the notation in terms of the dependency between x and y. Subsequently, they were asked: "Do you identify this symbolism with a process?"

> DB: Well if you see just $y(x)$ that really doesn't mean anything. Usually you see an equal sign and an equation following it.
>
> LAF: It depends on what's on the other side of the equal sign.

The responses seem to be consistent with the fact that neither student associated the notation with a rule to follow to find the output. The interviewer dug a bit further with the question: "If I just write $y(x)$, would you identify a process with it?"

> DB: I guess I would because I know that it has to go through some process to get the value of y.
>
> LAF: No.

Notice that DB identifies the output with y only, not with $y(x)$. LAF is definite in not identifying the notation with a process. Both are uncomfortable with associating the notation with a process. DB exhibits an image of the notation as having meaning only if the process is defined. Next the interviewer probed their interpretation of the notation as an output of a function. Each was asked: "Do you identify that symbolism $y(x)$ with output?"

DB: Yes. y is definitely the output.

LAF: Yeah.

While both DB and LAF interpret the notation as function notation, they demonstrate some conceptual difficulties. They find it difficult to accept $y(x)$ alone without setting it equal to an algebraic expression that describes the process. Thompson suggests that "the predominant image evoked in a student by the word "function" is of two written expressions separated by an equal sign." (Thompson, p. 24) Both students illustrate this image. LAF is unwilling to attach a process meaning to the notation until the process is identified, suggesting a procedural orientation to the notation. Both accept the notation as representative of output, but DB seems to be attaching the output to y only, rather than $y(x)$. The connection between input and output appears tenuous.

The interviewer chose to delve further by providing two specific examples. First, both were asked: "If I say $y(x) = 4$ does this represent a process, an output, or both to you?"

DB: That would be an output because you're not giving the equation you are just giving basically $y = 4$ when x is input.

LAF: Both. It's more of an output though.

Both students were uncomfortable with the given statement. Student difficulty with constant functions is well-documented. (See Markovits *et al.* and Tall & Bakar, for example.) Both students considered 4 was an output. DB believes there is some process performed on x to obtain 4. She sees the notation as a specific ordered pair rather than as a general statement of a function. LAF exhibits more flexibility, being willing to consider the statement as a process.

The interviewer followed with a question about a more prototypical function: "If $y(x) = 3x - 7$, does this represent a process, an output, or both?"

DB: That's a process because now you are giving the equation as to how you would establish the value of y.

DB is much more comfortable with this more common, prototypical form. DB demonstrates little flexibility in shifting between process and output.

LAF: Both.

LAF is very definite about the answer. The interviewer asked LAF: "How is it an output?"

LAF: Because that whole expression will give you that single number answer for output.

LAF was asked: "How is it a process?"

LAF: You have to substitute your input for x to solve for y.

LAF appears to have a more robust image of the notation. She is less procedural and views the notation more flexibly.

Another question asked on the post-course survey was: "Assume that f is the name of a function. Is there a difference between $3f(2)$ and $2f(3)$?" Nineteen (31%) of the respondents said "yes" while 30 (48%) said "no." The remaining students either did not reply or wrote something that could not be classified. LAF and DB responded quite differently. LAF simply wrote "no" indicating she did not see a difference between the two. DB wrote: "Yes, the value of the number in the parentheses is the independent variable which would affect the value of the function." DB's grasp of the notation at the time of the survey appears better than LAF's. Let's look at how they responded to the same question in an interview setting.

> DB: Different the way I would read this since f is name of function. The 3 and 2 in parentheses are input. How I am reading this is that I would find the value of f for the given input and then multiply that output by the number in front of f.

DB's interpretation is consistent with her written answer. LAF has changed her mind when asked the same question in the interview.

> LAF: (Shakes her head) Different because if this is a function then the numbers in parentheses are your inputs and they are different and then the outputs are different and I don't think they are related.

LAF's physical demonstration (shaking her head) suggests she remembers answering this question on the written survey. Both students seem to understand the meaning of the notation. Both were subsequently asked to find both $3f(2)$ and $2f(3)$ given a table for f, an equation for f, a function machine for f, and a graph for f. DB responded correctly to the question for all four representations. LAF initially had trouble finding $3f(2)$ given a table. She was confused about $f(2)$. The interviewer prompted her by noting that 2 was the input and that $f(2)$ was the output. With that hint she did the problems flawlessly.

Conclusion

The student population for this research is a high-risk group who have had little prior success with mathematics. Using "function" as a focal point of their beginning algebra course, the authors hoped to provide them with a vehicle to build meaning into their work with algebra. A large percentage were able to correctly recognize $f(x)$ as function notation and differentiate the notation from the notation for multiplication. However, only a small percentage could differentiate between $3f(2)$ and $2f(3)$. The two students interviewed, DB and LAF, demonstrated varying strengths and weaknesses in interpreting the notation. LAF appears to be able to interpret the notation more flexibly than DB, but DB demonstrates a better grasp of the relationship between input and output implied in the notation. Neither is at a stage where they see $f(x)$ as an object, while LAF seems to be on the verge of being able to flexibly see the notation as a process. While both the surveyed students and the interviewed students demonstrated some common misunderstandings about function notation, the data suggests that the notation is not beyond the conceptual grasp of students at this level. The analysis of this data will be used to influence future curricular design in order to help students confront some of the problems in their concept images of the notation.

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INVESTIGATING EXPONENTIAL FUNCTIONS: THE ROLE OF TABLES

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The study of functions is both a major organizing theme for secondary mathematics and a critical prelude to the study of calculus. In this study, through a set of teaching interviews, students enrolled in a pre-calculus college course investigated a set of contextual problems in exponential growth and decay. The students had access to both a graphing calculator and a computer-based, multi-representational analytic tool. The evidence from these interviews suggests that the use of tables (in a spreadsheet-like environment) provided a crucial link in the students' transition from a 'multiply and add' strategy to a multiplicative view of the functional relationship. The students' use of both the graphing calculator and the computer environment are discussed. Conclusions regarding limitations in the current technology and suggestions for curriculum development are given.

Introduction

Within the last three years, hand-held graphing calculators have rapidly become affordable technology for the mathematics and science classroom. While there is an extensive body of research on the use of the four-function arithmetic calculator among elementary school-age children, relatively little research has been done on the effectiveness of the hand-held graphics calculator for student learning and for its impact on student problem-solving skills (Dunham, 1995). Weaknesses in students' understanding of the function concept after conventional instruction have been well documented. Multi-representational, computer-based tools appear to have some positive impact on students' understandings and on their problem solving skills (Confrey & Doerr, 1996). The purpose of this study is to investigate students' conceptual understanding of exponential functions, using graphing calculators in conjunction with a computer-based multi-representational tool.

Theoretical Background

This paper will address student strategies in investigating exponential growth and decay through contextual problems. This study is part of a larger research project on examining students' conceptions of growth and rate of change in technology enhanced environments. In this paper, I present an analysis of the development of a population growth model, which includes the formation of conjectures, qualitative reasoning about those conjectures, and the validation of one or more conjectures. The validation of the conjectures is seen as a negotiated process that involves the interactions between the students, their understanding of the original problem situation and its constraints, and the use of computing and calculating technology. This view of student's conceptual development builds on earlier research that posits a non-linear, cyclic approach to model building (Doerr, in press).

The role of multiplicative strategies, as described by Confrey and Smith (1994), provides a theoretical basis for examining student understandings of the exponential function. In their work, these researchers argue that "splitting" is an equally strong approach to understanding multiplication as is repeated addition. The multiplication world of the exponential function has its origins in the actions of splitting, which is a distinctly different operational view of multiplication than is repeated addition. This would suggest that seeing the constancy of successive ratios should be as essential to an understanding of exponential functions as is the constancy of first order differences in linear functions. However, the exponential functions are not quite so simple in this regards. In particular, the covariation of quantities is a mixed model in that the independent variable will vary additively while the dependent variable changes multiplicatively. In this study, I examine how students move to a multiplicative view of the exponential function. The students used the Function Probe software for symbolic, graphical, and tabular analysis of the data and a hand-held graphing calculator.

Data Sources and Analysis

Ten pairs of college students, enrolled in a pre-calculus course participated in an hour and a half teaching interview. The students volunteered to participate in the interviews and their performance in their pre-calculus class reflected a wide range of achievement. The students were familiar with both the software and their calculators from their course work throughout the semester. Each interview session was video-taped, audio-taped and transcribed for analysis. In addition, the hand-written work and computer work done by each pair of students were collected for subsequent analysis. The students were free to use both their calculators and the computer software throughout the problem-solving session.

In their course work, the students had previously investigated problems of bacteria growth, where the growth factor was always two, since the populations doubled in a fixed time period. Each session began with the following problem:

> The population of Townville in 1860 was 1500 people. The population grew at about a rate of 2.8% per year until 1960. What was the population of Townville in 1940? When did/will the population exceed 20,000 people? Write an expression that can be used to find the population at any time t in years. In 1960, the growth rate slowed down to 1.6% per year. According to these figures, what is the estimated population of Townville in 1990? in 2000?

Upon completion of the first problem, the students then investigated a structurally similar problem in exponential decay:

> The original price of a jacket was $120. The price of the jacket is reduced by 10% each week until the jacket is sold. Assuming no one buys it earlier, when will the price of the jacket be $60? $50? How long will it take for the price of the jacket to decrease to $1? When is the price of the jacket decreasing most rapidly? If the original price of the jacket were $80, how would your graph, table and/or equation change? If the weekly discount were 15%, how would your graph, table and/or equation change?

This second problem represents a significant extension of the exponential model developed in the first problem situation.

Results

Each pair of students was successful in solving the posed problems. I will present in detail the three critical episodes of the strategy taken by one pair of students in investigating the first problem situation and the two critical episodes for the second problem situation. The first episode begins with the students' reformulation of the posed problem. The students recognize some similarity between this problem and the bacteria growth problem they had worked on in class. They recognize that an essential difference between this problem and the bacteria problem is that the population is not doubling, but rather increasing by a percentage. They move directly to the problem of finding a way to calculate the population in any given year without computing all the populations in between. They hypothesize that an expression for this might be P=1500 + (0.028)^y, where y is the number of years since 1860. This is rejected when they calculate the population in 80 years to be the same as the current population:

S2: We are looking for 1940, so from 1860 to 1940 is 80 years. So it would be to the 80th

I: Uhum.

S2: I don't know what kind of a number that will give us.

S1: This one right here.

[pause - 20 seconds]

S2: 1,500.

I: Something is wrong there.

S2: That wouldn't be possible. That equals 0. Yeah.

I: You did .028 to the 80th. Right. That would make that 0.

[pause - 15 seconds]

S2: That can't be .028 then.

Student One suggests that they look at intervals of fewer years. Student Two observes that the equation would be the same, but concurs that this might nonetheless be a useful strategy. The students then use a "multiply and add" strategy to calculate the population for each of the first three years.

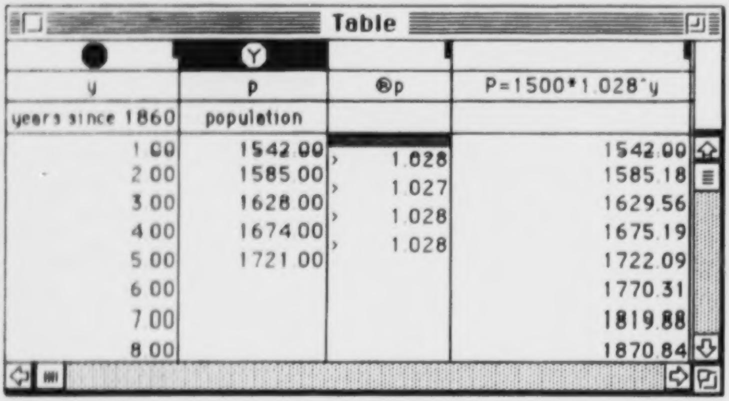
The second episode begins as the students explore this multiply and add strategy. They enter the population for the first three years into a Function Probe data table (see Figure 1) and hypothesize a new relationship: P=1500*(0.028)^y. They explore the multiply and add strategy:

S2: Like the first one was 1,542. So the first one works out because you are multiplying it by 1,500. But then on the second one, you want to multiply it by 1,542, not 1,500 do you know what I mean? Because after it increases you have to multiply the new population by the .028.

S1: Are you saying that the next time we probably will have the 1,585 or something because we're going on to the next one. Do you know what I mean?

S2: We are trying to get it through 1,585.

The second conjecture is then rejected as it only provided a correct value for the population when $y = 1$. But clearly the problem for the students has shifted. They recognize the recursive relationship wherein the new population depends on the current population but that understanding does not translate readily into a symbolic form that will allow them to calculate the population in any given year.



y	p	®p	P=1500*1.028^y
years since 1860	population		
1.00	1542.00	› 1.028	1542.00
2.00	1585.00	› 1.027	1585.18
3.00	1628.00	› 1.028	1629.56
4.00	1674.00	› 1.028	1675.19
5.00	1721.00		1722.09
6.00			1770.31
7.00			1819.88
8.00			1870.84

Figure 1. Successive ratios for the population growth

The third episode begins as Student Two suggests looking at successive ratios for the table entries for the population. She sees that the ratio of 1.028 is "how you get it to add." Note that initially the successive ratios appeared as 1.03 since the default for the software is set at 2 decimal places.

S2: That's [the ratio] constant. That is weird because it is not the percent or anything, it is the rate.

S1: We times it by .028. I don't know why 1.03 [unintel]

[pause - 14 seconds]

S2: Oh, you know why? Because it is the one, the whole, plus the .028 which they just rounded to .03!

I: Okay. Can we see?

S2: Because you have to add the whole, the 1,500.

I: So that is the 1 and then it is.

S2: It's a hundred, it's like 128 percent. It is like 102.8, that is how you get it to add. Okay.

They then conjecture that the population would be given by P = 1500*(1.028)^y which is confirmed by their first five years of population data (see Figure 1). From this, they were able to quickly answer all of the questions in the posed problem.

The students then turn their attention to the jacket problem. In the first episode, the students identify the necessary operations. Student One suggests that the operation might be subtraction or division to reduce the value of the jacket and that they should proceed year by year:

S1: But uh, you would be reducing it. So maybe subtracting? I'm not sure.

I: Okay.

S1: Because it is saying reduce by 10% each week. So maybe if we divided or....

I: Okay.

S1: Or subtract, somewhere in that ball park to reduce the number.

S1: Um [pause - 8 seconds] Um, I remember I, like in this one, I said something about like you know try it year by year. And it kind of worked. You know, it worked.

Student Two immediately recognizes the similarity to the previous problem and generalizes:

S2: Would it be 120 times point

S1: .10

S2: It is reduced by, would it be times .9 because you don't add it. You're subtracting it.

S1: Yeah.

I: That is what we just talked about.

S2: So 100% minus 10%. So it would be one whole minus .1, .9, wouldn't it?

I: Uhum.

S2: So it would be 120 times .9?

I: And what does that give you?

S1: 108. Well, that is the same thing as multiplying 120 times .10 and then subtracting. It gives you 12 and then it gives you, you know. So you can do it both ways.

They then use their calculators to compute the first four sale prices and enter them into their Function Probe table.

The second episode begins as Student Two conjectures the relationship P=120*.9^W, where W is the week of the sale. Student One is still uncertain about the use of subtraction or division, but their conjectured relationship agrees with their calculated values. They then go on to answer the remaining questions posed in the original problem situation.

Discussion and Conclusions

This pair of students began their investigation by shifting to the problem of finding the population in any given year without finding the intermediate populations. The students used their calculators in a multiply and add strategy to calculate the new population from the old for each of the first five years. They posed and rejected two symbolic relationships which used the percentage increase as the base of the exponential function. Their shift to a multiplicative relationship came with the move to the table environment, where they examined successive ratios and found the relationship between the base of the exponential function and the percentage increase. The students were then able to successfully generalize this relationship in a decay problem situation. Neither student showed any inclination to examine the graphs of the data for either problem.

The table environment was a crucial link in moving from a "multiply and add" strategy to a multiplicative strategy for expressing the total population as a function of time. Throughout their investigation, the students used their calculators to explore and confirm patterns in the population growth. The students used their calculators to quickly test some early hypotheses and to generate population values in which they had confidence. It was the table environment, rather than the graphical of view of the relationship, that motivated and confirmed their symbolic conjecture about the functional relationship. There was limited evidence from the interviews that the students understood the significance of the value of the growth factor. Neither the graphing calculator nor the software environment supported the symbolic expression of the recursive relationship that the students saw. The lack of tables in the graphing calculator was a limiting factor in their use.

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SITUATIONS AND FUNCTIONS

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This paper, which reports part of a larger study about high school students' understanding of functions, explores one student's sensemaking activities involving quadratic functions. It contrasts the student's success with problems that differed by the extent to which she needed to attend to the situational context and thus reason about quantities. The paper suggests that students conceptions of situations and quantities are factors in students' reasoning about everyday problems and supports separating numeric from quantitative reasoning. Finally, it suggests that these factors must be taken into account if we are to understand students' reasoning about mathematical concepts.

The use of realistic situations and applications is integral to the mathematics reform movement (NCTM, 1989) and forms the basis for many curriculum projects. Reasoning about everyday situations includes reasoning about quantities, yet many studies about functions at the high school level are only indirectly connected to how students reason about quantities in situations. In addition, since research indicates that situations interact with students' mathematical reasoning, their use is debated. See, for example, (Boaler, 1993; Janvier, 1981; Silver et al., 1993, Klein et al., 1996). This paper explores reasoning about functions in situated problems.

Understanding students' reasoning about functions in everyday problems involves understanding students' conceptions and representations of both situations and quantity, and their connections to mathematical concepts. Silver (1993) has developed a semantic processing model to characterize "situation-based reasoning." He distinguishes between the situation itself, the problem about a situation as described by a piece of text, and the student's mathematical representations of the problem. Kaput (1995), Thompson (1994), and Resnick and Greeno (quoted in Resnick, 1992) separate quantitative reasoning from numerical reasoning. Thompson defines a *quantity* as a conceptual entity, and a *quantitative operation* as nonnumerical and associated with the comprehension of a situation, whereas *numerical operations* are those that are used to evaluate a quantity. Research based on the distinction between quantity and number indicates that students sometimes reason directly about quantities in problems without recourse to quantification (Irwin, 1996; Resnick, 1992; Thompson, 1994). Distinguishing between students' conceptions of situational models and mathematical representations of situations, as well as between quantity and number, can be explored by noting where students attention lies during problem solving. This paper contrasts one students' success on problems that involved similar mathematical functions between the same variables but where problem tasks differed by the ease with which one could shift from reasoning about quantities to reasoning about number or mathematical representations.

Data and Method

Data reported here are part of a larger study investigating what sense students made of functions during a summer instructional program that had functions and modeling as central concepts (Fey & Heid, 1989), introduced all functions through situations, and provided rich experiences with multiple representations. The research focused on whether and how students' conceptions of functions were tied to specific mathematical (function class) or representational contexts (tables, graphs, symbolic, or verbal descriptions of situations). During data gathering, the definition of situations evolved from a representation class to a kind of problem context that provided quantities for the problem. The method of data collection and analysis for this study was consistent with the interpretive traditions of research (Erickson, 1986) and case study methods (Merriam, 1988). Data were collected over a 6-week period and consisted of interviews with four focal students in a class that was part of a Regional Math and Science Center for the Upward Bound Program, along with pre- and posttests, written homework, and fieldnotes and videotapes of classroom observations. This paper focuses on work by Toni, one of the focal students.

Results and Discussion

During interview problems, students often reasoned about quantities and quantitative relationships without evaluation. For example, Toni could fluidly fit a graph to verbal descriptions of functional relationships between two quantities without reference to any numbers. In addition, students relied heavily on the everyday situation itself to provide support for problem solving. For example, in a problem about students' travel mode and their distance from school, Toni placed one student at the point (40,2) on a graph of distance in relation to time, rather than at (20,2), solely on the sensibleness of the pair of numbers: if one walked 2 miles to school, 40 minutes was more sensible than 20 minutes. The situational context of other problems, however, invoked informal knowledge that confounded, rather than assisted, problem solving. Below Toni's work with two situated problems, where profit was a quadratic function of ticket price, is contrasted.

Conflict between Informal Model and the Problem Given

The interview problem in Figure 1 was given after 18 hours of classroom instruction that included exploring shapes of graphs, patterns of change in data tables, and connections between horizontal and vertical translations of graphs for quadratic functions. Students had worked with many examples of functions involving profit, costs, attendance, and ticket price for school events.

Carl has been assigned to come up with the price to charge for tickets for a concert the senior class is using for a fundraiser. From previous years, Carl has learned that if they hire a celebrity emcee, the function relating ticket price and profit was

$P(t) = -100t^2 + 700t - 200.$

Carl reasoned that if they didn't hire the emcee, they could save his $200 fee. His friend Beth said he should reconsider the problem because he would need to lower the ticket price by at least 3 dollars to get anyone to come if no emcee was hired. She told him he could consider this new situation by using the function

$P(t) = -100(t - 3)^2 + 700(t - 3)$

Was she correct? Which plan should he follow?

Does the size of the auditorium make any difference?

Figure 1. The Profit Problem

That the problem was about an everyday situation affected Toni's problem solving. She reasoned about quantities in the situation using a model that conflicted with those in the problem and which made it difficult for her to correctly interpret the functions and their representations. Even though several times she determined for herself that t was ticket price, Toni repeatedly forgot whether the variable t stood for tickets or ticket price. At one point after she became stuck, I judged the problem task too complex and narrowed the focus to the first function rule. Toni continued to identify $700t$ as profit and 200 as the money to pay the emcee, and she introduced the possibility that $-100t^2$ was a cost for producing tickets. She recognized that in one profit model, costs are subtracted from something, but seemed to confuse profit, $700t$ in her model, with income. Her reference to $-100t^2$ probably refers to a simulation done in class.[1] In any case, Toni associated terms or coefficients of the polynomial with quantities from the situation. She defined a profit (income) model as both price * tickets and as income - costs. Her work here is in contrast to that with many other problems where function rules were given in symbolic form and where she readily evaluated a function for a specific value for the independent variable. Toni's attempts here to give each term or coefficient a separate meaning as a situation quantity surely is suggested by the 200 in the problem statement itself, however, Toni seemed unable to proceed because of her inability to reconcile the function given with her informal understanding of the quantities, profit, costs, and income.

Function Dependency

Part of understanding functions is knowing that a function rule represents a dependency between two quantities. That the problem was situated also contributed to Toni's difficulty understanding the function rule as a relationship or dependency between two variables, ticket price and profit. As can be seen below, because of the difficulties she was having, I tried and repeatedly failed to get Toni to accept the function rule as a given relationship that could be interpreted independently from its specific situational origin:

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[1] Paper page limits prohibit displaying data to support these conclusions

G: Are you trying to figure out where this function came from? Or where the numbers in the function came from?

T: Yeah.

G: Ok. Why don't you just assume that they have a long history at this school for making, for having these shows and they know that this function rule is what the profit will be if they're given the ticket price and they hire an emcee for 200 dollars, ok? They know that. That's just information that's given to them. Ok? So, how, can you find out anything about profit from, um, from any ticket price?

T: [T points to the top formula] Could this maybe be from the year before?

G: But like the whole formula could be from the year before. Not the individual pieces but maybe they keep a file to help the people the next year and they say this is the formula you should use if you hire an emcee for 200 dollars. Ok, so if like a ticket costs 10 dollars, if they charged 10 dollars, can you tell me what kind of profit they might make? [points to top rule] If they hire an emcee for 200 dollars and if they charge 10 dollars for a ticket, do you know how much profit they will make?

T: You have to know how many people are going to be there and how much the tickets cost to them to--

G: You have to know how many people are going to be there?

T: To know the profit that they're going to make.

Twice I had tried to explain where the rule might have come from and indicated its generation could be ignored. I tried to shift the task to simply using the rule by suggesting first that she consider a single ticket price and then later a specific ticket price. However, Toni continued to reason about quantitative operations and remained uncertain about how to interpret the rule numerically. Her insistence that the rule must refer to the past and that the number of people coming, that is, the number of tickets sold, must be known before you can compute profit indicates she was unable to consider this function as a valid relationship between profit and ticket price. She might have been thinking of it as a formula where values she associated with computing profit must be known before profit could be computed or might have understood profit as a quantity that could not be computed a priori an event. In either case, her conceptions of the quantities inhibited her ability to evaluate them.

In contrast, on problem 9 of both the pre- and posttest, Toni interpreted correctly the profit for a specific price, the maximum profit possible, and the ticket price that would yield a maximum profit for *profit from a rock concert as a quadratic function of ticket price*. (The function was given graphically.) On the posttest (only) problem 13 she correctly answered similar questions about *profit for a school play as a function of ticket price* given in symbolic form, $P(t) = -50t^2 + 800t - 1950$. Why might this be? In the pre-/posttest problems, students could momentarily suspend the situation, attend to the mathematical representation and a procedure for interpreting numbers within it, and return to the situation only to interpret the mathematics, that is, students could quickly move to numeric operations and need not reason about quantitative operations. To solve the interview problem, however, one had to attend to quantities because the problem task required it. By the time the interview task was narrowed to one function rule, Toni's disposition to sensemaking and understanding of the quantities and quantitative relationships made it almost impossible for her to shift to numeric operations.

In conclusion, this study supports those models of students' reasoning about situations that separates quantitative from numeric reasoning. It suggests that theories of students' reasoning about functions separate students' conceptions of situations from their conceptions of the problem and mathematical representations of the problem. Further, it suggests that these conceptions are factors in students' reasoning about mathematical concepts.

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CHANGE IN TEACHING ABOUT FUNCTIONS: CONTENT CONCEPTIONS AND CURRICULUM REFORM

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A high school teacher's flexible and comprehensive mathematical content conceptions supported his implementation of an innovative curricular approach to functions. The teacher (Mr. Allen) was studied for two years, and this paper focuses primarily on year two findings about the subtle but meaningful changes in Mr. Allen's conceptions and instruction evidenced as he gained comfort with the new curriculum. In particular, the paper illustrates how Mr. Allen revised his pedagogical content conceptions through a complex interaction of his mathematical content conceptions and classroom experiences with students. These results extend what it means for conceptions and instruction to be interrelated, particularly in the context of instructional reform in mathematics.

Based on the vision of the current reform movement in mathematics education (e.g., National Council of Teachers of Mathematics [NCTM], 1989), numerous innovative curriculum development projects have emerged (e.g., Connected Mathematics Project, Core-Plus Mathematics Project, Mathematics in Context). The changes called for in the implementation of such materials place substantial demands on experienced teachers to modify both the content and the strategies of mathematics instruction. In light of the significant challenges teachers face to modify existing routines and practices, it is crucial that we investigate how teachers deal with such calls for reform. There is a growing body of empirical literature supporting the notion that teachers' conceptions critically influence what goes on in the classroom (Fennema & Franke, 1992; Thompson, 1992), but research evidence about the role of teachers' conceptions in making the transition to reformed practice is still quite limited. Nonetheless, several recent reports suggest that teachers' mathematical and pedagogical conceptions may profoundly affect their ability to deal with innovative curricula (Gamoran, 1994; Wilson, 1990).

This study focuses on conceptions of mathematical functions. The reform movement's envisioned treatment of the function concept in grades 9-12 includes modeling real-world situations using functions, multiple representations and interpretations of relationships, translations between different representations, and recognition of the variety of problem situations that can be modeled by the same type of function (NCTM, 1989). These recommendations contrast with the traditional secondary curriculum's approach that focuses largely on symbolic manipulations and isolated representations of de-contextualized functional relationships. Few studies have investigated teachers' thinking about functions, and those that have done so tend to focus on prospective teachers (Even, 1993; Wilson, 1994) or elementary teachers who are not subject specialists (Stein, Baxter, & Leinhardt, 1990). How do experienced secondary teachers make sense of current reform recommendations for the teaching and learning of functions?

This paper explicates one theme from an in-depth case study (Merriam, 1991) of the role of a high school teacher's content conceptions in his implementation of the materials of the Core-Plus Mathematics Project (CPMP) over a two-year period. The present article extends reports of first year results (Lloyd & Wilson, 1995, in press) to consider the following question with particular focus on year two of the study: How did the teacher's conceptions of curricular content (e.g., functions) and instructional experiences surrounding that content impact each other under the influence of reform?

Research Design

The subject of this interpretive case study is Mr. Allen, a high school mathematics teacher in a small urban community in the Northeast. Mr. Allen had largely adhered to traditional practices during his 14 years of teaching, but he voluntarily implemented the CPMP materials in one class of ninth grade students during the 1994-95 school year and again in 1995-96. The other four courses that Mr. Allen taught each year were components of the school's ongoing traditional curriculum. The CPMP program, funded by the National Science Foundation, is field-testing materials for a four-year high school mathematics curriculum. CPMP lessons emphasize the modeling of real-world situations as a way to explore mathematical concepts and incorporate numerous features of the TI-82 graphics calculator. The CPMP unit of focus for this study, *Patterns of Change*, treats functions as dependence relationships using student-centered, contextualized activities with particular focus on exploring and connecting multiple representations of relationships.

In year one (1994-95), a series of eight interviews focused on Mr. Allen's conceptions and teaching practices, and I observed his CPMP classroom daily while he implemented the *Patterns of Change* unit (26 days). In year two (1995-96), I conducted eight interviews and observed Mr. Allen's CPMP classroom while he implemented the *Patterns of Change* (21 days) and *Linear Models* (22 days) units. During both years, I observed selected groups of students, took fieldnotes during all classroom observations, and made photocopies of written classroom artifacts. Observations of *Patterns of Change* were video-recorded, with a cordless microphone carried by the teacher. I audio-recorded and transcribed all interviews.

Ongoing analysis over the two-year course of the study allowed me to refine and develop questions, methods, and themes to detect subtle differences in Mr. Allen's first and second years of CPMP implementation. I maintained a comprehensive case study database by compiling and reviewing transcripts, fieldnotes, classroom artifacts, audiotapes, and videotapes to create accounts of interviews and lessons. The case study database enabled repeated scans of the data to identify major categories and patterns and facilitate searches for contradictory examples (LeCompte & Preissle, 1993). I developed major categories and patterns using Spradley's (1979) taxonomic and thematic methods of analysis, and over time, synthesized themes within and across different data sources.

Results

During his two years of CPMP implementation, Mr. Allen made similar content emphases during his interactions with students. However, as the results described below exemplify, there were subtle differences in his instruction over time that offer meaningful insights into his process of learning to teach with the new curriculum.

First year reports (Lloyd & Wilson, 1995, in press) offer detailed explications of Mr. Allen's conceptions of functions and how they influenced his first year CPMP instruction. Mr. Allen's conceptions included comprehensive understandings of a variety of relationships and real-world examples of functions, and tight connections between different families and representations of functions. Although he also had set-theoretic conceptions of functions, his thinking about particular relationships was dominated by covariation or dependence notions (e.g., "As one variable changes, what happens to the other?") and graphical displays (which assisted him in easily "seeing" patterns). For Mr. Allen, it was of utmost importance to understand covariation patterns underlying particular functions.

In his year one *Patterns of Change* instruction, Mr. Allen placed emphasis on dependence relationships by repeatedly engaging students in discussions framed by the same questions that guided his own thinking about functions: "*Is* there a relationship?" and "*How* are the variables related?" Mr. Allen also demonstrated a high regard for the centrality of the variety of tables, graphs, equations, and verbal descriptions in the *Patterns of Change* activities because of the different information that each representation provides. However, Mr. Allen frequently gave precedence to graphs (his personal preference) by portraying them to students as optimal displays of patterns. As he pointed out to one student, "The table gives you times and heights, but the graph gives you *the relationship* between time and height." He also added "investigative graphing" tasks to assignments and urged students to make effective use of the graphics calculators to create fruitful visual representations. He capitalized on graphs in his instruction as a starting point from which to stress important connections between different representations of the same situation, and to accentuate the features that distinguish different families of functions. In sum, Mr. Allen applied his graphical strengths and covariation understandings to create opportunities for students to "see" features of a variety of functional situations.

In Mr. Allen's second year teaching with *Patterns of Change*, it became even more evident that, to him, understanding functions meant understanding covariation relationships. He consistently drew attention to the unit's focus on "different patterns of change" and "*how* things are changing." Mr. Allen used the terms *function* and *variable* more explicitly and with greater frequency in his discussions with students in the second year, for example during an investigation of how a theater's daily income relates to the number of tickets sold:

> There is a *relationship* between income and tickets sold. They are using the word *function*. What is a function? Simply, a function is when you take a look at a couple of things and you start trying to look for patterns.

Accordingly, Mr. Allen portrayed variables as "things that you might relate" and, for most problem situations, he drew attention to the variables and to the notion that one "is a function of" the other. He further emphasized functions as relationships by bringing in new examples such as the following:

> If you go to Pine Knob and you buy tickets and the tickets are $15, is there a relationship between the number of tickets you buy and how much money they're going to charge you? That's a function. A function is a relationship.

That Mr. Allen frequently brought in examples from outside the CPMP materials in year two, but rarely did so in year one, reflects not only his increased emphasis on functions as covariation patterns, but also his growing personalization and comfort with the curriculum.

A related example of a subtle but meaningful difference in Mr. Allen's year two instruction involved his use of explorations of multiple representations to lead students toward detailed understandings of covariation patterns. As in year one, he communicated and demonstrated his valuation of student construction and exploration of multiple representations of relationships. He portrayed different representations as important because of personal preferences and contextual considerations, for instance, "Sometimes people can't think of an equation but they can say it in words," and "If you have several things that you can take a look at, you may be able to use one in one case, and in another case use something else." In contrast to year one, Mr. Allen made greater attempts (and was more successful in those attempts) to treat different representations equitably. In general, his year two instruction emphasized equations, tables, graphs, and verbal descriptions, with no one representation missing out significantly.

However, Mr. Allen occasionally gave preference to certain formats during class discussions of particular problems. Often in these cases, students reminded Mr. Allen of the usefulness of the neglected representations. For example, consider the following transcription of Mr. Allen's interaction with two students about a relationship presented in both graphical and tabular forms:

> Megan: What is a rule?
>
> Mr. Allen: They want you to come up with a rule, an explanation of how you would figure it out. So is there a way that you could put into words or is there a way, whatever you think, of how this pattern is occurring on the graph?
>
> Toby: Is it as the price increases by 2, the customers decrease by 5?
>
> Mr. Allen: What do you think about that? *[Looking at Megan]*
>
> Megan: I agree with him.
>
> Mr. Allen: Is that kind of the pattern that's occurring there? *[Points to graph]*
>
> Megan: Yeah.
>
> Toby: You don't have to look at this necessarily *[Points to graph]*. You can look over

> here *[Points to the table]*.
>
> Mr. Allen: Right. Some people like to see a graph, some people can figure it out from a table.

Toby's recognition of the pattern in the table directly challenged Mr. Allen's graphical tendency which he conceded was a personal preference that students may not necessarily share.

Despite his increased classroom treatment of multiple representations, Mr. Allen communicated to me that he maintained a personal preference for graphs ("Visually I get a better sense of what's going on with the relationship because I can see it over an interval") rather than tables ("I have a harder time when I'm just given data and I have to get the relationship going between just the pure numbers"). In Mr. Allen's opinion, much of his struggle with tables stemmed from his years of experience with the approach of the traditional curriculum which emphasizes translating "an equation of a line into a graph without having to make a table." Although he didn't "have a lot of practice taking a look at tables," his CPMP experiences, particularly with the graphics calculator, helped him to better appreciate the power of less visual representations such as tables:

> With a graph it's visual and you can see the pattern, but you don't necessarily have right in front of you the actual pairings of the data [as with a table], and to be able to look critically at --this is an x with a y and this is a new x with a y.... With a graph you can see that it's a line and there's a constant increase, but you don't necessarily see exactly the data that might be able to tell you exactly a specific slope or a rate of change.

Given the fact that tables were neglected in many of his discussions with students in the first year instruction of *Patterns of Change*, the above statement is particularly noteworthy for its indication of Mr. Allen's emerging view that tables can at times provide more specific and useful information than a graph.

Discussion

The significance of the results presented above lies in the subtle differences in Mr. Allen's two years of CPMP instruction. During both years, he communicated and demonstrated similar goals for student learning about functions: recognizing and describing the nature of relationships between two changing things, and understanding that relationships can be represented in multiple ways which provide different details about the nature of the relationship. Although these goals were evidenced in both years, he enacted these goals more explicitly and more frequently in year two. Accordingly, Mr. Allen expressed during a year two interview that "I'm doing a better job at ... trying to focus in on the idea that a function is just a relationship and there's many ways to take a look at it." Mr. Allen's perception of his improved ability to "focus in on the idea" clearly emerged from his year of experience: his increased familiarity with the materials enabled and motivated him to make more explicit those concepts and ideas that he determined to be most important for students.

As a combined result of his classroom experiences with CPMP and his mathematical conceptions, Mr. Allen displayed evidence of having developed new pedagogical content conceptions. For example, although his conceptions of functions were largely graphically-centered, their integrated and comprehensive nature along with his classroom focus on understanding covariation patterns contributed to his ability and motivation to learn from classroom experiences why multiple representations are important to students' understandings. This result illustrates how strong and flexible mathematical content conceptions can play a pivotal role in facilitating the development of new pedagogical content conceptions when classroom experiences invite or demand it. Further, Mr. Allen's case extends our understanding of what it means for conceptions and instruction to be interrelated, particularly in the context of reform. Use of the CPMP curriculum enabled Mr. Allen to think about mathematics and pedagogy while teaching, and thus learn from his classroom practices.

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FUNCTION UNDERSTANDING IN A GRAPHING APPROACH CURRICULUM

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The function concept is of fundamental importance in college mathematics, but it is also one for which students seldom develop a satisfactory understanding. One possible explanation of the difficulties that students have with the function concept is that the students may develop only naive conceptions of functions. Researchers hypothesize that students acquire the function concept in two stages: *procedural* in which the function is viewed as a process such as assigning values, and then *structural* in which the function is an object on which operations can be performed (Sfard, 1991). The transition from procedural to structural (reification) is very difficult for students. It is not surprising that Kieran (1992) concludes from examining the findings of algebra learning research that "the majority of students do not acquire any real sense of the structural aspects of algebra." However, the situation is not hopeless. Kieran further cites studies showing that students can develop structural conceptions of certain algebraic notions if provided with experiences involving technology. She maintains that "the one (research area) that seems to stand out as being clearly in need of attention is that of finding ways to develop structural conceptions in students."

This quasi-experimental semester- long study examined the effects of a graphing-approach college algebra curriculum along with the TI-82 graphing calculator on students understanding of the function concept and traditional algebraic skill. Four classes at a large state university served as the population with the balanced design: two teachers, each teaching one experimental and one control class. The classes studied the same topics, but the traditional curriculum focused more on hand calculations. The questions explored in this study were framed in the process/object theory. Of interest in this study were students' ability to interpret, model real world situations with, translate among representations of, and to reify functions, as well as students' overall understanding of the function concept. A pre- and posttest expressly for analyzing function understanding was used. Questions within these four categories were analyzed using a pretest and posttest specifically designed by Dr. Brian O'Callaghan at Louisiana State for analyzing function understanding.

Findings indicate that students in the graphing-approach classes had a better understanding of functions as a group than did the traditional algebra group. They had significantly higher scores on the function test and on all four subtests. No significant differences were found on the departmental final exam which measures traditional algebraic skill without a graphing calculator.

Current theory suggests that technology has the potential to help facilitate the development of structural conceptions of functions. This study supports that theory and suggests that the graphing calculator has some effect in bridging the gap between procedural and structural conceptions.

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GEOMETRIC THINKING

REPRESENTING, CONNECTING AND RESTRUCTURING KNOWLEDGE: A MICRO-GENETIC ANALYSIS OF A CHILD'S LEARNING IN AN OPEN-ENDED TASK INVOLVING PERIMETER, PATHS AND POLYGONS

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We observed children constructing several rectangles and triangles satisfying a given perimeter. Here we describe the nature of one child's knowledge of length by examining active and reflective engagement with a constructive task involving length as a component of perimeter. We analyze knowledge structures of both number and space by building and testing a model of how one child applied mathematical length concepts in a problem setting involving the formation of various polygons; we analyze her engagement with a task calling for multiple instances of rectangles and triangles satisfying a specific perimeter. Finally, we consider the validity and the predictive power of this model for furthering our understanding of how children represent, connect and restructure knowledge from specific domains such as number and space.

National level curricular standards recommend connecting diverse knowledge structures and applying concepts across several contexts (NCTM, 1989), yet there is little research analyzing the substance of such connections. Given tasks that demand the integration of knowledge from differing domains, what strategies do children evince as they work to coordinate and restructure their own understanding? We describe complex changes in children's knowledge structures of both number and two-dimensional space. We argue that children forge connections between diverse representational contexts as they integrate knowledge structures from multiple domains (Steffe & Cobb, 1988). Finally, we describe the qualities of the consequent knowledge structures.

Background

There are many accounts of the nature of children's development of complex knowledge, but few descriptions relating the development of domain specific knowledge of two-dimensional length (Clements & Battista, 1992). Recent investigations of the micro-genetic growth of children's mathematical knowledge suggest that such growth involves several interconnected schemes and that new knowledge develops unevenly with respect to previously reified conceptual elements of these schemes (Schoenfeld, Smith & Arcavi, 1993). Our research accepts and supports this perspective.

Methodology

Motivation for specific focus of our study stems from our previous research that suggests a need for more detailed models of cognitive strategies linking the geometric and number domains (Clements, Battista, Sarama, Swaminathan & McMillen, in press). We interviewed 28 students from ages 8 to 16 years. Here, we focus on changes in the mathematical knowledge and understanding of one 8-year-old student, Mandi, as she explored a particular aspect of the relation of number and space (variation of shape and the additive composition of perimeter) during a 30-minute interview. In previous research (Clements et al., in press), three strategies were indicated for third grade children investigating length and operating within two-dimensional space. This research suggests a need for a transitional stage between the second and third types of strategy.

The interview was conducted in keeping with a moderately structured guideline adapted from research on proof and mathematical reasoning (K. Koedinger, personal communication, January 1994). We redesigned the guideline to focus on knowledge of space and number; we subjected the instrument itself to several rounds of trial, evaluation and revision. Consequently, the interviewer focused on specific aspects of the children's knowledge and reasoning, at once following emergent themes as well as specific research questions developed in earlier studies (Clements et al.). Further, the interviewer intervened to support students' knowledge if they were not familiar with concepts critical to this investigation (e.g., to explain the term "perimeter" if they did not know its definition). The interviews ranged between 26 minutes and 55 minutes in duration.

A 48 cm long plastic straw divided evenly into two-centimeter segments was given to the children. Notches indicated two-centimeter segments along its length. The following written introduction was given to each child:

> The Perimeter Problem
>
> Introduction- Finding rectangles and triangles with a perimeter of 24
>
> 1. Explore the set of rectangles with a perimeter (total length of all sides) of 24. Draw and label several rectangles.
>
> 2. Describe the whole set of rectangles with this perimeter. How many are there altogether? Can you explain your answer? Can you prove that you found all of the rectangles having a perimeter of 24?

Findings

Describing the length of the sides of a polygon

Mandi drew a rectangular figure while looking at a rectangle-like straw figure she had formed. When asked about the length around the straw figure, she answered 24. When the interviewer asked her about the length of its sides she paused for some time. She looked puzzled. She explained that she was trying to ". . . figure out how big the sides are." Eventually, the interviewer suggested that she think about the straw pieces she had just been counting as the units for measuring the sides. Instantly, she picked up the straw and formed the figure again. She counted along its sides, marked each side with the number "4," counted the top and finally marked both the top and the base with the number "6" (the diagram had actual dimensions of 6.5, 11.3, 7.0, and 11.2 cm). She explained that the sum of these four sides would be 20 rather than 24.

We take her pause to indicate a disconnection between the whole perimeter and its constituent parts. We expect that she would have begun counting the sides directly if she had forged a part-whole connection for the sides the perimeter. Instead, she cast around for a starting point to measure the sides. At the same time however, her rejection of 20 as the sum of the sides of her diagram suggests she did connect the numbers 6, 4, 6 and 4 with the perimeter number, 24. We note that she did not describe the 4-6-4-6 diagram as a newly discovered rectangle as other third-grade children did; she saw it as a mistaken record of the straw rectangle, mistaken because its sides did not sum to 24.

Expecting the sum of the lengths of the sides to be the length of the unbent straw

Later, Mandi counted along one side, the top of the straw figure, and said, "Eight." She wrote the number eight along both the top side and the base of her diagram. Then she said the four sides summed to 23, pointing to the 8, 4, 8 and 4 labeling her figure. She seemed alarmed and ready to count each side of the straw figure all over again. The interviewer suggested that she recalculate the sum, at which point she found 24. Here again we find evidence for her belief that the straw figure represented a composite of sides summing to 24. She immediately questioned her count of the four sides. Once again she rejected a sum other than 24, suggesting that she expected the diagrammed rectangle to have sides that summed to the overall length of the now-bent straw. Further, by returning to count the unit pieces composing each of the four sides and then taking the sum of those four sides she demonstrates her belief that the unit pieces along the sides *compose* the perimeter.

Constructing triangles from number triples

After being prompted to look for more triangles, Mandi drew a fourth triangle with a similar shape. This diagram was the smallest yet. She labeled it with a base of 4 and sides of 10, labels that were consistent with its indicated lengths. Mandi focused on the number 24 as a unit of units. She broke the number 24 into components, each of which she was able to break again into sub-components while maintaining a sense of the original whole being the number 24:

Inter. Tell me how you got that triangle with those labels.

Mandi: Ten and fourteen would equal twenty-four…and five and five would equal ten…, and adding fourteen is twenty-four.

Inter. So, here are three triangles [with the perimeter of twenty-four]. Are there more?

Mandi: I don't think so.

Inter. Can you tell me altogether, how many there are possible? This is the really important part of the whole question. Can you find some others?

Mandi: [Lips moving, appears tense, but still silent]

Mandi: I'm trying to think- - one of my ideas is two tens and a four.

Inter Does that work?
Mandi [Drawing a ten-ten-four triangle, of similar shape as before, isosceles]
Inter How did you think of that?
Mandi Two tens and a four make 24.

Mandi broke apart the number 24 into three parts over and again. When given the option of using the straw or working on paper, she abandoned the straw and began generating possible number triples for triangles. She did not check these by forming triangles with the straw, nor did she make a scaled drawing of these triangle "ideas"; had she drawn these diagrams to scale she might have noticed the "impossible" combinations of side lengths. We take this to indicate that she was not operating on a spatial image of a straw figure bent into three parts corresponding to these numbers. In sum, she operated on numbers and combinations of numbers that were not connected to spatial images.

Fixing a violation of the Triangle Inequality Theorem

The interviewer asked Mandi whether she could use the *ruler* to draw the same triangle, using centimeters in order to fit the drawing on one standard sheet of paper. She produced a long line segment (measuring approximately 14 cm) with a couple of short line segments sticking up along one end and then asserted that it could not be done. When pressed whether it was impossible, Mandi suggested that one might draw the triangle if you "had 14 and then 5 and 5 inches." This statement was puzzling to the interviewer and so he asked her to demonstrate. She drew the long base with the cm scale, 14 cm and then flipped to the alternate ruling system for inches and completed the top two sides of a triangle measuring 5 inches each. Thus we believe Mandi's concept of perimeter was based on a part-whole understanding of perimeter since she purposefully used disparate units of length to address a violation of the Triangle Inequality Theorem; her solution depended on a smaller unit yielding a smaller length overall (one side of length 14) in combination with a larger unit yielding a larger length overall (two sides of length 5).

Discussion

When Mandi was first asked to describe the length of the sides of the 5-7-4-8 straw quadrilateral, she paused for several moments and finally said she was trying to see how big the sides were. Her lengthy pause was symptomatic of a less-developed connection between Mandi's algebraic knowledge and her geometric knowledge about rectangles. Mandi dissociated perceptual parts from the perceptual whole; she was puzzled when asked to describe the lengths of the individual sides even though she had been quick to describe the perimeter of the whole figure. Had she operated on an abstracted unit of units, she would have described the length of the sides in terms of unit pieces (Steffe, 1991).

We also note her facility with arithmetic operations on numbers of the magnitude necessary to answer this kind of question; 24 was not such a large number that she would struggle to carry out arithmetic operations involving it or numbers that constitute it. Thus, she might have reasoned that the straw length could be segmented into its constituent parts. In subsequent portions of this interview she did just that, inventing several number-triples that summed to 24. Because she was capable of segmenting and dividing numbers, and because she exhibited part-whole understanding of numbers composing a sum of 24, we argue that her slow and labored response to the challenge to describe the lengths of the individual sides followed from her less-developed knowledge of *spatial* part-whole relations.

Alternatively, Mandi may not have understood perimeter to constitute a length at all; instead, her definition of perimeter may have been entirely distinct from length. However, we reject this perspective because it is inconsistent with several instances of Mandi's work in which she associated perimeter with the composite length of the sides of polygons she diagrammed. For the five triangles that she attempted, Mandi always chose three numbers that summed to 24, the whole length around the figure. Likewise, when the sum of four sides of the 6-4-6-4 diagram did not match the 24 from the straw figure she concluded that she counted wrong. She expected that a correct count for each of the sides of the straw rectangle would yield 24. This too suggests that her sense of the proper relation of parts and wholes was strongest in the realm of number and that it was less developed in the realm of conceptual space.

Given her level of development, we would expect her to conserve length. Conservation of length around a bent path would require mentally partitioning the straw into three or four sides and further, into unit pieces. We note further that some of the third graders observed in earlier research were thought to exhibit such a sophisticated understanding of how a polygon might represent a bent *path*. Mandi conserved the length of the straw when she monitored the sum of 6, 4, 6 and 4, expecting 24, and again when she rejected 23 as the sum for 8, 4 and 8 and 4. This kind of understanding of paths and perimeter for polygons was only rarely observed among third grade children (Clements et al., in press). Such conservation of length suggests a developing grasp of geometric operations involving deformations, bending a line segment into a closed "path" around a triangle.

Implications

Teachers need to develop instructional sequences that support students shifting attention between two domains that they expect students to integrate. Students may need opportunities to "break" rules in one domain while following a conjecture in a second domain, and then return to the same conjecture afterwards and reconsider the effect of ignoring that first domain as it bears on the situation. Mandi demonstrated that such a partial focus is sometimes productive and yet her subsequent struggle to monitor those attempts at triangles that were not geometrically tractable suggests that instructional sequences that allow students to isolate one domain will also require extensive teacher scaffolding.

Students will need substantial support to work past the negative instances that can follow from the loss of balance between two integrated domains. Research on positive versus negative examples suggests that teachers must determine whether certain irrelevant features of a given concept are salient (Clements & Battista, 1992, p. 449). If so, then a mixture of positive and negative examples should follow; if not, then positive examples should predominate.

Mandi's struggle in manipulating the straw may have led her to a practical approach: She used the straw to represent ideas, yet without expecting to replicate the image of any particular shape such as the rectangle. Her flexible approach to representing ideal shapes using "less-than-ideal" material is unremarkable for such a young child since elementary school aged children must continually cope with challenges beyond their level of manual dexterity. Such an approach is instructive, though, for designing instruction at this age level: Such students may not take manipulatives as sources of empirical information but rather as expressive media for their images and ideas.

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DEVELOPMENT OF STUDENTS' SPATIAL THINKING IN A CURRICULUM UNIT ON GEOMETRIC MOTIONS AND AREA

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We investigated the application and development of spatial thinking in an instructional unit on area and motions, part of a large-scale curriculum development project funded by the National Science Foundation. We also investigated the role of noncomputer and computer interactions in that development. We collected data from paper-and-pencil assessments and case studies as we pilot tested the unit with two third graders and field tested it in three third-grade classrooms. Results revealed strong positive effects on spatial abilities and the establishment of spatial-numeric connections; they also provided information about students' strategies for solving the unit's spatial and spatial-numeric problems. A distinguishing feature of these strategies was the degree to which students applied a unitizing operation to construct spatial and numerical units and "units of units."

Our research had three purposes: 1.) to chart the development of students' spatial competencies as they engaged in an innovative geometry unit concerned with geometric motions and area; 2.) to determine whether there were gender differences in spatial abilities in this context; and 3.) to document students' problem solving and reasoning as they worked on the activities in the unit.

The instructional unit, which engages third-graders in a series of combined geometric and arithmetic investigations, was developed as part of a large-scale curriculum development project, funded by the National Science Foundation (NSF), that emphasizes meaningful mathematical problems and depth of exploration of, rather than mere exposure to, content. The philosophical basis for the curriculum (and our research) is constructivism, with its emphasis on developing problem-solving abilities based on rich conceptual knowledge structures and students' invention of strategies (Clements & Battista, 1990). A major objective in designing the activities of this unit was to develop students' spatial abilities both because such abilities are valuable in themselves (National Council of Teachers of Mathematics, 1989) and because specific spatial abilities appear to be related to mathematical competencies (Brown & Wheatley, 1989; Clements & Battista, 1992; Fennema & Carpenter, 1981; Wheatley, Brown, & Solano, 1994). We accept the theoretical position that mathematical understanding is constructed to a large extent in images, much of which are spatial in nature.

Another objective of our study was to chart students' use and development of concepts and strategies as they engaged in the nontraditional activities contained in the unit. Such activities provide a context that may reveal new aspects of students' mathematics thinking.

Descriptions of these aspects can also guide teachers and curriculum developers in designing and implementing activities with similar goals. In particular, the instructional unit emphasized the following mathematical goals: knowledge of and ability to communicate about area, motions, and congruence; construction of two-dimensional space; construction of and operation on arithmetic and spatial units; connection of spatial and numeric ideas; and development of spatial competencies such as spatial visualization. Given that the unit was based on a constructivist theoretical framework, assessing attainment of these goals also provides one test of the practicality of this framework for curriculum design and teaching.

Data indicating the importance of students' construction of and operation on arithmetic and spatial units emerged during a pilot study. Both theoretical and empirical work has illustrated the significance of the construction and use of units, and especially superordinate units (i.e., "units of units") to students' mathematical development, in such diverse areas as arithmetic meanings and strategies (Steffe & Cobb, 1988) and spatial and geometric concepts (Clements & Battista, 1992; Wheatley & Cobb, 1990). The activity of constructing unities is a mental operation of segmenting of experiences, isolating one aspect of experience while at the same time leaving it embedded in the whole. Uniting a collection of items into a whole "lifts" its structure to a higher mental plane. The result of such reflective abstraction is a structure, a composite unit, consisting of individual abstract units (Steffe & Cobb, 1988). In pursuing our third research goal, we found the unitizing construct to be critical in understanding the strategies students used to solve the combined spatial-numeric tasks that the curriculum presented.

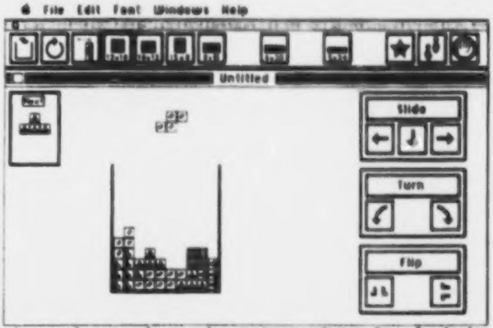
Method

Procedure. Flips, turns, and area (Clements, Russell, Tierney, Battista, & Meredith, 1995) was taught to third graders in two different situations. In the first, a pilot test, two of the authors taught the unit to two students. The second situation, a field test, involved one of the authors teaching the unit to a third-grade class (Class A) the following autumn. During the following winter, two teachers taught a revision of the unit to their third-grade students (classrooms B and C). This provided a field test in which the regular classroom teachers, rather than an author of the curriculum, conducted the activities.

We conducted videotaped case studies with both pilot test students. These raw data were transcribed and analyzed qualitatively. Data collection for the field tests involved observations of the whole class and small groups of students as they worked on the activities. In addition, in class A only (classrooms B and C could administer only that paper-and-pencil testing already mandated by the school district; the administration had a policy against "over-testing"), we administered the Wheatley Spatial Ability Test (WSAT, Wheatley, 1978) as a pre- and posttest measure of spatial competence. On each of the 100 items on this test, students are presented with several congruent geometric figures. They must decide whether each shape would have to be flipped and rotated or just rotated in order to be superimposed on a target shape (see Appendix). We calculated both a correctness score and proportion correct score. The correctness score was calculated by subtracting one-half of the number of incorrect responses from the number of correct responses. This provides a measure of competence corrected for guessing. The proportion correct was the number correct divided by the number attempted. This indicates whether any change is due to the ability to solve more items (within the test's time limit) versus the ability to respond more accurately to items. Retesting with the WSAT has revealed only slight increases; thus, substantive increases might be attributed to the treatment.

Participants. Participants in the pilot test were two volunteers from a suburb of Buffalo, Ryan and David, both 9 years of age. Participants for the field tests were 23 students in one suburban (Class A) and two urban third-grade classes (Classes B and C). Students are named in this article by their class letter and a researcher-assigned student number. Each class had approximately the same number of boys and girls.

Curriculum and software. Flips, turns and area (Clements et al., 1995), is designed to develop third-grade students' concepts of geometric motions and area as well as their spatial-visualization abilities by having students move, subdivide, and combine shapes. Students use manipulatives, paper, and a computer activity to explore tetrominoes and area concepts. The computer activity, *Tumbling Tetrominoes*™, is a modified version of the popular video game, *Tetris*™. The software has the following features.



1) Emphasizes ideas about area (player covers a region; rows do not disappear as in Tetris™; player's score is the area of the region he/she covered) and geometric motions (player must select a motion, Slide, Flip, or Turn, first, and then the "direction" of the motion). 2) De-emphasizes speed (the tetrominoes do not descend continuously). 3) Includes tetrominoes designed to highlight the tetromino unit and its orientation. 4) Allows replacement. 5) Allows replaying the same game. 6) Includes a "next box." 7) Includes rectangles of different dimensions. 8) Provides challenges designed to elucidate motions. If player chooses the Star tool, he/she can play a game at the "star level." This more

challenging level does not allow player to erase the placement of the last tetromino, nor does it allow flips. 9) Allows stepping through a game. 10) "My rectangle." Player can choose any rectangle that fits on the screen.

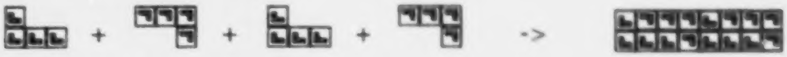
Results and Discussion

The first research goal was to chart the development of students' spatial competencies as they engaged in a unit on motions and area. Results of the Wheatley Spatial Abilities Test (WSAT) indicate that the activities had a strong positive effect on one spatial competency, the transformation of internalized images. Given that spatial abilities such as these are related to mathematical competencies (Brown & Wheatley, 1989; Clements & Battista, 1992; Wheatley et al., 1994), these results imply that the curriculum and software described here may make a significant contribution to students' mathematical development.

The second goal was to document the problem solving of students as they work on the activities. In general, the tasks and the basic settings were motivating and meaningful to the students, and allowed a variety of approaches for students of different abilities. Further, students' behaviors yielded useful information about their thinking in the context of specific activities. As reported in other research (Clements & Battista, 1992), students initially included orientation as a property of figures without reflecting on its appropriateness. Discriminating when orientation is important, and doing spatial transformations necessary for that discrimination, are important mathematical capabilities. Students began describing the motions needed to prove congruence when they needed to convince a peer that two shapes were congruent.

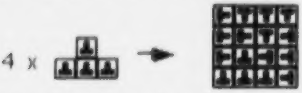
Perhaps the most important finding regarding the second goal was the significance of "units" and "units of units" thinking in student's solving of both spatial and numerical problems. Three types of strategies were identified. The first involved simple iteration or trial-and-error processes on singletons that they did not consider higher-order units. For example, in constructing spatial patterns, students filled by trial-and-error or filling in from the sides of the rectangle. As an example in the numerical domain, students might count single squares or apply arithmetic to groups of single squares that are not conceptualized as units.

To use strategies of the second type, students construct and operate on units (students conceptualize each unit simultaneously as being constituted of multiple singletons and as being one higher-order unit). For example, in constructing spatial patterns, students continued a pattern of tetrominoes that led to a "good covering," but did not coordinate superordinate units (units of units).



In the numerical domain, students counted units of units, such as groups of ten, but exhibited the use of additive rather than multiplicative processes.

To use strategies of the third type, students coordinate units, building and applying units of units, or superordinate units. For example, in constructing spatial patterns, students extended their patterning activity to create a tiling with a new unit shape—a unit of unit of unit squares that they recognized and consciously constructed.



In the numerical domain, students used fully multiplicative processes.

Given the research describing the difficulties many students of this age have with arrays, area concepts, and area models for multiplication (Battista & Clements, 1996; Carpenter, Coburn, Reys, & Wilson, 1975; Clements, Sarama, & Battista, in press; Outhred & Mitchelmore, 1992) the spontaneous invention of pre-multiplication (e.g., counting by twos and tens) and multiplication concepts (e.g., counting the number of tetrominoes, then multiplying by four; or multiplying the number of rows and columns) suggest that the activities provided students with worthwhile mathematical experiences. Perhaps most important, the curriculum provided a coherent set of activities that engaged students in unitizing, in both the geometric and numerical domains, each to the benefit of the other.

The determination of scores motivated students not only to stretch their competencies with mental and other forms of computation, but also to improve their strategies in the *Tumbling Tetrominoes* computer game. The students who initially left gaps while waiting for a specific tetromino, became more flexible in their thinking. Thus, the learning of the various concepts appeared to be synergistic. As another example, the students became familiar with the tetrominoes and how geometric motions affected them. As they manipulated the tetrominoes, found and replicated patterns, and created units of units with them, they built mental images of the shapes, their features, and their interrelationships. This setting, therefore, may be superior to other spatial tasks (e.g., jigsaw puzzles) because intimate knowledge of the tetrominoes promoted the synergistic combination of (a) dynamic imagery and (b) reflective knowledge of the properties of geometric figures. A final example of synergy was the creation of different shaped rectangles, in which students simultaneously constructed rectangles and evaluated the area or number of squares in each new rectangle. For all students observed, the game provided both a meaningful setting and a strong motivation for such arithmetic activity.

In general, the activities motivated and aided students in building more sophisticated and systematic strategies. Inflexibility and unfamiliarity with composing shapes to create other shapes limits students' spatial thinking. Without experiences with such compositions, they would be limited in a wide range of tasks, such as solving geometric problems requiring disembedding and using array models for multidigit multiplication. Activities such as the ones in the unit investigated here may make substantive contributions in developing such capabilities.

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WHY CAN'T FOURTH GRADERS CALCULATE THE AREA OF A RECTANGLE?

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One hundred seventy-seven children in grades 4, 5, 7, and 9 were individually interviewed to find out (a) when they use the formula (length x width) taught in 3rd and 4th grade and (b) whether their difficulty in using the formula is due to a confusion between area and perimeter or to their unidimensional (linear) thinking as explained by Piaget. The children were shown two rectangles (4" x 5" and 7" x 3") photocopied on a sheet and asked to measure them with a ruler to decide which one would have more to eat if they were chocolate bars. The great majority of the children did not use the formula before 7th grade, and their difficulty was found not to be due to a simple confusion between area and perimeter.

According to many educators including those involved in the National Assessment of Educational Progress (NAEP) (Lindquist & Kouba, 1989), children cannot calculate the area of a rectangle because they confuse area and perimeter. NAEP reported that only about 3% of the 3rd graders and 40% of the 7th graders chose the answer of 30 when presented with a rectangle having a width of 6 and a length of 5. When the same rectangle was presented with 30 squares drawn in it (6 across and 5 vertically), the percentages choosing the correct answer were not much better. Only about 14% of the 3rd graders and 54% of the 7th graders chose the answer of 30. (The percentages cited are of all students taking the test.)

Length x width is usually taught in 3rd and 4th grade and reviewed in 5th grade. In this instruction, children are first asked to cover a rectangle with many squares and to count them. They are then asked for a way to calculate the number of square units. This calculation is often included in a chapter on multiplication.

Piaget (Piaget, Inhelder, & Szeminska, 1948/1960) argued that children's difficulty in accepting the formula (length x width) is due to their inability to understand how two *lines* produce an *area*. He pointed out that if we give squares to children, calculating the area by multiplying is easy for them because the squares are discrete objects. Given two lengths of 5 and 6 cm (two continuous quantities that are unidimensional), however, the calculation of an area requires "reducing the area to an infinite set of line infinitesimally close to one another" (p. 350). Children cannot think about a line as "an infinite series of neighboring points" before the age of 12 (Piaget & Inhelder, 1948/1967). Likewise, "length x width" does not become intelligible until it is understood that the area itself is reducible to lines, because a two-dimensional continuum amounts to an uninterrupted matrix of one-dimensional continua" (Piaget, Inhelder, & Szeminska, 1948, 1969, p. 350).

To find out if Piaget's explanation was more adequate that the usual one stating that children confuse area and perimeter, the following questions were investigated:

1. When do children use the formula (length x width) taught in 3rd and 4th grade (ages 8-10)?
2. Is children's difficulty in using the formula due to their unidimensional (linear) thinking or to a confusion between area and perimeter?

Method

One hundred seventy-seven children in grades 4, 5, 7, and 9 were individually interviewed and videotaped in three suburban schools near Birmingham, Alabama. Each child was shown a sheet with two rectangle photocopies on it (4" x 5" and 7" x 3"). The question posed was: "If these were chocolate bars, and I asked you to choose the bigger one that has more to eat, which one would you choose?" Regardless of the child's response, he or she was then given a ruler and asked to use it "to make sure you are right," etc.

If the child did not compare 4" x 5" with 7" x 3", he or she was given a Color Tile (1" x 1"), then 25 color tiles, to find out if it was possible to elicit a comparison that came close to 4 x 5 vs. 7 x 3.

Results

Table 1

How Children Compare the Amounts of Two Chocolate Bars (in Percent)

	4th gr	5th gr.	7th gr.	9th gr.
	n=36	n=43	n=35	n=39
Ages	9-10	10-11	12-13	14-15
Length x width	12	18	60	61
(Could explain formula)			(26)	(25)
(Could not explain formula)			(34)	(35)
Unidimensional (linear) comparisons				
20" > 18" (perimeters)	23	29	20	8
10" > 9" (length + width)	25	21	14	8
7" > 5" (two longer sides)	17	5	0	6
	(65)	(55)	(34)	(22)
Others	20	28	6	17

As can be seen in Table 1, the percentage using the formula (length x width) was negligible in 4th grade (12%) and increased only to 18%, 60%, and 61%, respectively, in grades 5, 7, and 9. This study thus supports Piaget's statement that , before 12 years of age, children do not think of using the formula that they supposedly learned in 3rd and 4th grade (ages 8-10).

Some children at each grade level based their numerical comparison on the perimeters (23%, 29%, 20%, and 8%, respectively) and said that the rectangle with a perimeter of 20" was more to eat that the one with a perimeter of 18". However, two other ways of using linear measurements were found. One was to *add* the length and the width and to say that 7" + 3" was more to eat that 4" + 5" (25%, 21%, 14%, and 8%, respectively). These percentages are very similar to those related to the use of perimeters. The other way was to compare only the longer sides of the rectangle with a length of (7" and 5") and to say that the rectangle with a length of 7" was more to eat (17%, 5%, 0%, and 6%).

The last category, "Others," consisted mostly of unsuccessful attempts to mentally superimpose one rectangle on the other by partitioning one of them. This spatial approach was two-dimensional and used linear measurement but was inexact and resulted in errors.

It can thus be seen in Table 1 that the three kinds of unidimensional (linear) comparisons decreased (from 65% in 4th grade to 55%, 34%, and 22%) as the two-dimensional judgments (length x width) increased. It was therefore concluded that children's use of perimeters to compare sizes is only one of the manifestations of their unidimensional (linear) thinking. In other words, children are not just confusing area and perimeter. To assimilate the formula taught in 3rd and 4th grade, children have to make sense of the idea that one can get and *area* (which is two-dimensional) out of two *lines* (which are both unidimensional).

The difference between children's thinking about discrete objects (Color Tiles) and continuous quantities (length and area) became clear when the Color Tiles were offered to the children.

A related observation is that no one in 4th grade and only one child in 5th grade used the term "square inches," which had been taught. When asked, "Twenty *what* do you get when you multiply 4 inches by 5 inches?" almost all the 4th and 5th graders replied "20 squares," "20 blocks," or "20 inches." In 7th grade, 9% of the students began to use the term "square inches." The language of the 4th, 5th, and 7th graders thus expressed discrete of unidimensional quantities. These children did not say "square inches" because this idea did not correspond to their way of thinking.

Educational Implications

It seems appropriate to ask 4th graders how many squares are necessary to cover a rectangle. However, educators must be aware that there is a significant difference (a) between numbers of squares (discrete quantities) and an area (continuous quantities) and (b) between unidimensional space. Further research is necessary to better understand children's constructive process. Research is also necessary in classrooms to find better ways of teaching children how to compare areas that cannot be compared directly.

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PREPARING K-8 TEACHERS TO TEACH GEOMETRY

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This discussion group will work towards the promotion of coordinated research and the exchange of scientific information related to the teaching of geometry to preservice K-8 teachers. This session is intended as a follow-up to the geometry plenary session and related research reports and short orals.

This group will (a) summarize the geometry plenary session, (b) discuss how information gained from this conference can impact geometry courses for preservice K-8 teachers, and (c) identify significant research questions related to a geometry course for preservice K-8 teachers and appropriate methods to investigate these questions.

Individuals interested in participating in this discussion group are encouraged to bring to the session (a) concerns about their course and (b) researchable questions they are interested in investigating related to their course.

As a result of participating in this discussion group, individuals will (a) gain a shared perspective of the plenary session and (b) share ideas with other individuals interested in researching issues related to the teaching and learning of geometry (e.g., the role of technology in the teaching and learning of Geometry, Geometry as a component in teacher education programs). These shared ideas, questions, and methods may serve as a foundation for future collaborative efforts among participants.

DIDACTIC MATERIAL FOR THE TEACHING OF DEDUCTION AND DEMONSTRATION AT MIDDLE SUPERIOR LEVEL

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We need to hold the new curricular indications with actual cognitive conceptions, in this way one important task is to have work methodologies and concrete examples. In this poster we carry out 1) one structure of work methodology; 2) some examples where those methodologies were worked, and 3) teachers' beliefs about both.

We considered that the teaching of demonstration we are trying to initiate at middle superior level, based upon Euclidean Geometry, requires previous instruction when working in the construction of conjectures with the help of intuition, which is not previewed in the mentioned curricula.

Our efforts are focused to change this situation, thus we designed didactic materials which include Euclidean Geometry contents (for middle level). The aim was to develop a process of deduction, and occasionally of demonstration, backed on an empirical conception and in the work of the construction of conjectures. The topics of the materials were area of a triangle, heights, median, and geometric places, to mention some of them.

During the development of the experience, information was compiled and organized in order to formulate, through conjectures, the proposition that describes the phenomenon. This conjecture or proposition arrives to a condition that requires demonstration of the assortment or, if it is possible, to re-adequate it when contradictions are introduced. At this point, we tried to prove constructively the need and the relevant role of mathematical rigor emerging as a necessity to demonstrate the propositions constructed by conjectures, how it's meaning is related to the operating mathematics context, and how without all this rigor cannot become possible.

The next stage of this process has to do with the re-accommodation of the information pertaining to applications or generalizations of the previous conclusions. We dedicated a teacher's guide for each series of problems that accomplished a double task: on one hand, it enunciates the aims of series, and in particular those of the problems; it also contained some suggestions taking into consideration that students would utilize such materials and, in this way, teachers would become involved when following this process. And on the other hand, it provides relevant information for teachers in order to understand properly the problems of the series.

The experience was successful: 65 of the 67 teachers held the structure and believed that it is a suitable way to help the introduction to deduction and demonstration in high school geometry.

A STUDY ABOUT PERCEPTION OF POLYHEDRA BASED ON BIDIMENSIONAL SYMBOLIC REPRESENTATION

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The present work is a report of a Geometry workshop carried out with teachers working in middle superior level institutions. Developed activities were focused on polycubes and stellar polyhedra's constructions. We include, as an example, a workshop's activity.

Geometry is still an open field to research several perceptions of the individual, particularly the tridimensional perceptions that are unattended to in school mathematics. Many questions have been formulated in this sense, but today not all the answers have been found.

Several researchers point out that spatial perception is still an uncertain theme in school mathematics (Gaulin, 1985; Gaulin, Noeltig, Puchlska, 1983), even from the moment in which the individual must represent a tridimensional shape graphically or symbolically. Besides this, tridimensional perceptions are curricularly unattended; constructive abilities are developed restrictly; and our students are not properly informed about the course of recent mathematics research.

The purpose of the workshop was the construction of simple and compound polyhedra shapes that lead teachers to think about tridimensional constructive experiences furnished for students along basic and middle levels, through the use of a constructive method of non-traditional polyhedra (which we will describe as polyhedra weave), based on handling rectangular strips with triangles. With these strips students are able to construct pyramids, octahedra, icosahedra and the so-called deltahedra, which are bodies formed by equilateral triangle faces.

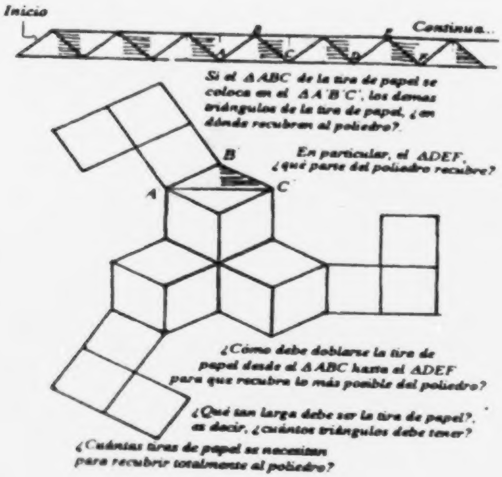
Activity 1. Covering a polycube order-4

> In this task we showed an order-4 polycube weaved in paper and gave participants a drawing representing the model. In this drawing, we asked them to answer a question linked to recovering such a polycube by means of rectangular strips of paper. Solutions may be represented in drawings or whichever symbolic representations participants would propose. We gave them paper for covering the polycube and hid it, in order for it not to serve as a constructive base.

For those who participated in the workshop, carrying out a task such as re-covering a deltahedron or a polycube, by means of rectangular strips, involves a series of difficulties that once they are surmounted, lead the individual to gain some advantages, such as representing objects, proposing temporal regularities, carrying out empirical experiences, and gaining more adequate knowledge.

Gaulin, C. (1985). *The need for emphasizing various graphical representations of 3-dimensional shapes and relations.*

Gaulin, C., Noelting, G. & Puchalska, E. (1983). *The communication of spatial information by means of coded orthogonal views.*



Si el ΔABC de la tira de papel se coloca en el $\Delta A'B'C'$, los demas triángulos de la tira de papel, ¿en dónde recubren al poliedro?

En particular, el ΔDEF, ¿qué parte del poliedro recubre?

¿Cómo debe doblarse la tira de papel desde el ΔABC hasta el ΔDEF para que recubra lo más posible del poliedro?

¿Qué tan larga debe ser la tira de papel?, es decir, ¿cuántos triángulos debe tener?

¿Cuántas tiras de papel se necesitan para recubrir totalmente al poliedro?

Figure 1. The problem of covering a polycube.

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PRESERVICE TEACHERS' IDEAS ON INVESTIGATIVE ACTIVITIES IN GEOMETRY USING MICROCOMPUTERS

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Given the dual role of microcomputers as both a computational and instructional tool, it is important to understand teachers' ideas concerning the use of technology as a tool for investigation (National Council of Teachers of Mathematics, 1989). In addition, with such a powerful opportunity for exploration, it is important as well to understand teachers' ideas on the nature of investigative activities within such a medium. As such, our study addresses the following questions: (i) How do preservice teachers design an investigative activity to teach a geometrical concept using microcomputers? (ii) What do these designs suggest concerning preservice teachers' ideas about the role of technology in instruction? (iii) What do these designs suggest concerning preservice teachers' ideas about the nature of an investigative activity?

Thirty preservice secondary teachers enrolled in the methods course, "Teaching Mathematics with Technology," at a large southeastern university participated in this investigation. Prior to this study, during a period of six weeks, all preservice teachers had completed a number of investigative/exploratory activities, participated in classroom discussions about the nature of investigative/exploratory activities, and individually developed such activities. The entire course was run in a computer laboratory and all students' work was done on the computers.

For this particular study, each student was given an assignment in which he or she was asked to develop an investigative activity using *Geometer's Sketchpad* to teach the concept of area of a regular polygon. Students were asked to include reflections on their activities as well. The tasks were completed as take-home assignments. The results of the qualitative analysis show a continuum of meaning among preservice teachers about the nature of investigative activities and the role technology plays in it.

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USING MICROSOFT WINDOWS LOGO TO TEACH CONCEPTS OF POLYGONS

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The presenter will post the e-mail address, which shows where the Microsoft Windows LOGO shareware may be found. The procedures for booting-up the Microsoft Works Logo software will be described on a handout created by the presenter. LOGO commands will be listed on a handout printed from the software help window. Instructions on how to enter and execute procedures will be described on a handout created by the presenter.

Instructions and explanations are given on two procedures, "Triangle" and "Square". The "Triangle" procedure contains the following statement: "Repeat 3[Forward 50; Right 120]". The "Forward 50" command instructs the turtle to move fifty spaces in the direction it is facing. The "Right 120" command instruct the turtle to turn right 120 degrees from the direction it is facing. This creates the 60 degree supplementary angle which will be an interior angle of the equilateral triangle. The "Repeat 3" command creates the three sides of the equilateral triangle.

The "Square" procedure contains the following statement: "Repeat 4[Forward 50; Right 90]". The "Forward 50" command instructs the turtle to move fifty spaces in the direction it is facing. The "Right 90" command instruct the turtle to turn right 90 degrees from the direction it is facing. This creates the 90 degree supplementary angle which will be an interior angle of the square. The "Repeat 4" command creates the four sides of the square.

The learner is encouraged to create "Pentagon", "Hexagon", and "Octagon" procedures. The "Pentagon" procedure should include the following statement: "Repeat 5[Forward 50; Right 72]", the "Hexagon" procedure: "Repeat 6[Forward 50; Right 60]", and the "Octagon" procedure: "Repeat 8[Forward 50; Right 45]".

The learner is also guided to discover a general procedure for polygons:

```
To Polygon :N :Size
        Repeat :N[Forward :Size, Right 360/:N]
End
```

Note: Sum of interior angles of a polygon (N is the number of sides): 180* (N-2)
Size of an interior angle of a polygon: [180 * (N-2)/N]
Size of supplementary angle to an interior angle: 180 - [180 * (N-2)/N] = 360 / N

The presenter requires students in a pre-service mathematics class for elementary teachers to create a design using LOGO. The presenter will display some of the pictures drawn by university students using LOGO.

PROBABILITY AND STATISTICS

CONNECTING STEM PLOTS TO HISTOGRAMS

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The goal of this study was to develop understanding of how students interpret stem plots and connect stem plots with histograms. In Fall 1994, 76 students in grade 6 and 51 students in grade 8 were post-tested (with open-ended questions) after an instructional unit designed to help them make connections between pairs of graphs. Students were asked to interpret the data in a stem plot and to create a histogram from the data in the stem plot. Although emphasizing the transition from stem plot to histogram appears to be a viable instructional strategy, students' understanding of representations using "grouped" data continues to be problematic, even after instruction.

This study was designed to develop understanding of how students interpret data presented in stem plots and then connect stem plots with histograms. Making connections between representations of mathematical concepts seems important for developing flexibility of understanding (e.g., Bowman, 1993; Janvier, 1987). In the area of graph comprehension, we are just beginning to understand the ways that connections between pairs of graphs might be developed and might be used by students (e.g., Bright & Friel, in press). Developing both mathematical and conceptual connections seems important in light of the emphasis by the National Council of Teachers of Mathematics (NCTM) on statistics as a content strand and on connections as a common thread across all content (NCTM, 1989). Helping students understand connections seems to be one way to deepen their understanding of graphs and their skill at using and interpreting graphs appropriately. The choice of a graph that "best" communicates characteristics of a particular data set must respond to the nature of those data. Instruction should help students understand the types of data for which particular graphs are most applicable.

The "reform view" of statistics demands that instruction be organized around critical inquiry rather than mastery of specified procedures. What makes a problem a statistical investigation is the way the question is posed, the nature of the data and the ways the data are collected, how the data are examined, and the types of interpretations made from such examination. A statistics problem, as outlined by Graham (1987), typically contains four components (a) question, (b) data collection, (c) analysis, and (d) interpretation, in some order. Kader and Perry (1994) suggest a fifth phase - namely, communicating the results of the interpretation.

Overview of the Study

During Fall 1994, students in grades 6 and 8 were asked to interpret information in stem plots and to use those interpretations to make histograms. The grade 6 students (N=76) were from three classes taught by one teacher, and the grade 8 students (N=51) were from three classes taught by another teacher in a different school district. Students were tested on stem plots and histograms after an instructional unit (Friel & Bright, 1995) developed specifically to highlight connections between pairs of graphs. The contexts used in the tests were scenarios believed to be familiar to students.

Stem plots and histograms seem "intuitively" to be connected; both representations are built on "groupings" of data. Stems usually represent the tens digits of the data values and leaves usually represent the ones digits. This means that the "20s stem" is a row in which data values from 20 to 29 can be placed. However, if 27 is the largest value less than 30, 27 will be the largest value that is actually displayed in the 20s stem. This sometimes causes confusion about the values that a stem represents and can make the transition to histogram somewhat confusing. It is theoretically easy to imagine rotating a stem plot 90° counterclockwise and visualizing a histogram drawn over the stems. However, in a situation like the one described above, if a learner perceives the data in the 20s stem as extending only from 20 to 27, then the intervals in the histogram might not be drawn (or labeled) with the same width. Too, stem plots sometimes visually obscure recognition of clusters of data. For example, if the data, 25, 29, 30, 31, are displayed in a stem plot with groupings by 10's, the value 29 appears closer to 25, since 29 is visually separated from the values 30 and 31.

Figure 1. Test Questions: Stem Plot to Histogram

Travel Time to School

Students were interested in how they used their time. They brainstormed a list of ways such as sleeping, eating, after school sports, and so on. Jim reminded them that some of their time is used just traveling back and forth to school. Some of the students thought this shouldn't count because it really wasn't much time at all. Others disagreed. The class wondered, "What is the typical time it takes to travel to school?"

Here is a stem-and-leaf plot they made to show data they collected about number of minutes it took them to travel to school in one day:

Minutes to Travel to School

0	3 3 5 7 8 9
1	0 1 3 5 6 6 8 9
2	0 1 3 3 3 5 5 8 8
3	0 5
4	5

1. How many students are in the class? How can you tell?
2. How many students took less than 15 minutes to travel to school? How can you tell?
3. Write down the three shortest travel times students took to get to school?
4. Write down the three longest travel times students took to get to school?
5. What is the typical time it takes for students to travel to school? Explain your answer.
6. Make a histogram to show the information about travel time that is displayed on the stem-and-leaf plot.

In the instructional unit, students were told about the conventions of histograms; that is, the bars touch, each bar represents data in an interval, an interval starts with the lower bound but does not include the upper bound (e.g., the interval 5-10 includes all data from 5 up to and not including 10), etc. This information was not, however, the central focus of the unit. The relationship between stem plots and histograms was explored visually by taking one set of data and representing it in stem plots based on 10's and 5's and then turning the stem plots 90° counterclockwise and "filling in" the bars (to cover the leaves) to create histograms.

The test questions on stem plots and histograms are shown in Figure 1. These items were part of a larger test. Since these representations were likely not fully familiar to students, it was important to probe their skill at reading information directly from the graphs. Students in this study were unfamiliar with stem plots prior to the instructional unit, so this part of the test was administered only after the instruction was completed.

Results

Grade 6. In grade 6, 64 (84%) of the students answered question 1 correctly; and 59 (78%) answered question 2 correctly. Virtually all of the explanations referred to "counting" numbers on the right of the plot (i.e., the "leaves"). Responses to questions 3 and 4 are shown in Table 1. For question 3, there was a clear split between two popular responses (41% correct for "3, 3, 5," and 32% for "3, 5, 7"), while for question 4, most of the students (80%) responded correctly.

Table 1. Numbers of Responses by Category for Questions 3 and 4: Grade 6

	Question 4						
Question 3	45, 35, 30	45, 35	45, 35, 28	40, 35, 30	7, 8, 9	other	Totals
3, 3, 5	31	2	0	1	0	1	35
3, 5	3	0	0	0	0	0	3
3, 5, 7	24	1	2	1	0	0	28
3, 3, 5, 7	1	0	0	0	0	0	1
3, 6, 7	1	0	0	0	0	0	1
3, 5, 6	1	0	0	0	0	0	1
0, 1, 3	0	0	0	0	1	0	1
1, 3, 5	0	0	0	0	2	1	3
other	0	0	0	0	0	3	3
Totals	61	3	2	2	3	5	76

For question 5, 52 (68%) of the students responded "23", and typically cited that value as the mode or most common. Of the 8 (11%) subjects who responded with intervals, 3 wrote "20-28," 2 wrote "20-23," 2 wrote "20-30," and 1 wrote "35-45." Of the remaining responses, 1 was blank, and 15 were scattered among ten different choices.

For question 6, 36 (47%) of the subjects created histograms numbering by 10's (11 correct, 25 incorrect), and 17 (22%) of the subjects created histograms numbering by 5's (4 correct, 13 incorrect). For the histograms numbered by 10's, students frequently mislabeled the intervals; a common mislabeling was 0, 1, 2, 3, 4, 5 (i.e., the numbers labeling the stems) rather than 0, 10, 20, 30, 40, 50. This incorrect labeling might be referred to as "transitional labeling" since it retains a particular aspect (i.e., the "stems") of the original stem plot. For the histograms numbered by 5's, students misplaced the data ending in 5 (e.g., 15, 25) and had trouble placing the 30, 35, and 45 in the proper bars. Of the remaining responses, 7 (9%) were blank, and 16 (21%) were not interpretable.

Grade 8. In grade 8, 44 (86%) of the students answered question 1 correctly; 43 (84%) answered question 2 correctly. Again, the explanations frequently referred to "counting" numbers on the right of the plot, though some subjects said they "added" the numbers on the right. Responses to questions 3 and 4 are shown in Table 2. There is still a split in responses to question 3, but the choice of "3, 5, 7" was less attractive for grade 8 students. Interestingly, the two grade 8 students (and the one grade 6 student) who responded "0, 1, 3" for question 3 responded "7, 8, 9" for question 4; these are, respectively, the three smallest and three largest leaves in the stem plot independent of the stems.

Table 2. Numbers of Responses by Category for Questions 3 and 4: Grade 8

	Question 4					
Question 3	45, 35, 30	45, 35	45, 35, 28	40, 35, 30	7, 8, 9	Totals
3, 3, 5	28	1	0	2	0	31
3, 5	2	1	0	0	0	3
3, 3, 3	1	0	0	0	0	1
3, 3	1	1	0	0	0	2
3	1	0	0	0	0	1
3, 5, 7	8	1	2	0	0	11
0, 1, 3	0	0	0	0	2	2
Totals	41	4	2	2	2	51

For question 5, 22 (43%) of the students responded "23" and cited that value as the mode or most, while 22 (43%) of the subjects responded with intervals. For these "interval responses," 11 gave intervals ending in 28 (e.g., 0-28, 16-28, 20-28), and 8 gave intervals ending in 30. These responses are consistent with the acknowledgment earlier that the visual separation between stems seems to overpower the possible recognition that the end of one stem might actually be "close to" the beginning of the next stem. Four students gave answers that explicitly or implicitly seemed related to the average of the data.

For the last question, 18 (35%) of the subjects created histograms numbering by 10's (9 correct, 9 incorrect), and 16 (31%) of the subjects created histograms numbering by 5's (3 correct, 13 incorrect). For the numbering by 10's, the most common mistake was labeling at least some of the intervals according to the range of actual data in a stem (e.g., 20-28) rather than keeping all of the intervals the same width. For the numbering by 5's, the most common mistake was a mis-placement of the value 45 in the 40-45 interval rather than the 45-50 interval. One student created a correct histogram numbered by 15's; 11 students left this item blank, five responses were uninterpretable.

Discussion

Across the responses, there is some evidence that students are "distracted" by the visual features of the stem plot. This is particularly evident in the grade 8 responses for question 5, for which students specified intervals that began and ended with the beginning and ending values of particular stems in the stem plot.

When students were asked to make a histogram, many tried to create histograms numbered by 5's rather than 10's. The stem plot already showed data grouped by 10's so it would seem almost "natural" to make the histogram by 10's. The students' decisions to use 5's may have been prompted in part by the fact that a histogram numbered by 5's would fit on the grid, and students may have thought they had to "fill" the grid. If students had been asked to draw their own grids, or if regular graph paper had been provided, the most popular choice might have been different. Their decisions may also have been influenced by the particular examples of histograms with data grouped by 5's and 10's in the instructional unit.

Students in grade 8 seemed more successful and responded in somewhat more sophisticated ways than students in grade 6. For example, students in grade 8 much more often gave intervals as responses to question 5 than students in grade 6. Interval responses seem more likely to capture the nature of "typicalness" than a single value such as the mode. Single values, like the mode, tend to be influenced by random fluctuations in data, whereas intervals would seem to be more stable. These differences across grades could reflect the developmental levels of the students, increased exposure to statistics instruction, or the impact of the particular teachers who taught the students. Since histograms are fairly abstract representations, with numerous conventions to be followed in their construction, we tend toward accepting the "developmental explanation" as at least part of the explanation of the differences across grades. Further evidence for this view comes from the data for question 6. In grade 6, 20% of the histograms were correct, and 21% were not interpretable. In grade 8, 25% were correct, which is slightly greater than the corresponding percentage for grade 6, but only 10% were not interpretable. In addition, the interpretable but incorrect responses by students in grade 8 seemed to contain fewer errors than the responses by students in grade 6.

The limited success of all students at creating histograms suggests that even after instruction, there are many aspects of histograms that are difficult for students to understand and use effectively. Indeed, there are many opportunities in the construction of any histogram for things to go wrong, and our scoring rubric for question 6 was correspondingly somewhat liberal in what was accepted as correct. For example, we accepted alternate placements for the labels on the x-axis.

The testing environment did not allow students unlimited time to "digest" the sense of the data in the stem plot, so it is conceivable that students focused more on the "form" of the stem plot and histogram rather than on the substance of the data represented in the stem plot. Students may have approached the task of creating the histogram as one of trying to remember histogram-construction procedures that they had been exposed to during instruction rather than as one of "sense making." The transitional labeling used by grade 6 students would seem to be evidence of this; that is, students tried to use the parts of the stem plot directly in the histogram rather than trying to transfer the information from the stem plot to the histogram.

We still have much to learn about students' understanding of stem plots and histograms and of the connections between these two graphs. These data begin to reveal some of the parameters of that understanding.

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THE INFLUENCE OF HANDS-ON ACTIVITIES ON STUDENTS' UNDERSTANDING OF SELECTED STATISTICAL CONCEPTS

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The use of investigative, hands-on activities is increasingly common in statistics education. The extent to which these activities facilitate students' understanding of statistics, however, is an open question. This research examines the effects of hands-on activities on students' understanding of selected statistical concepts. The data suggest that although hands-on activities facilitate the formation of correct or partially correct understandings, they can also promote the formation of misconceptions. Novices inability to discriminate between the salient and non-salient features of hands-on activities may be a source of these misconceptions.

Introduction

The use of "hands-on" activities - activities in which students conduct experiments, simulations, or surveys, collect data, analyze the results, and develop generalizations on the basis of their analyses - is increasingly common in statistics education. For example, contemporary classroom materials (Burrill, et. al., 1992; Spurrier, Edwards, & Thombs, 1995), journal articles (Hunt, 1994; Rossman & Short, 1995), and conference presentations (Zahn & Davis, 1995) all advocate the use of experimentation, simulation, and concrete manipulatives. In general, statistics educators have embraced the idea of allowing students to construct an understanding of statistical concepts through active experimentation and discovery.

Although the effectiveness of hands-on techniques is well-documented in the mathematics education literature (Friedman, 1978; Post, 1980, 1988), empirical support for hands-on statistics education is just beginning to emerge. For example, several recent studies (Garfield & delMas, 1989; Sullivan, 1995) report positive effects on students' conceptual understanding when traditional statistics curricula are restructured to focus on computer use and student exploration. There is some evidence, however, that the mere use of manipulatives and experimentation provides no guarantee that students will develop conceptual understandings. For example, Well, Pollatsek, & Boyce (1990) discovered that many students do not understand the effect of sample size on variability, even after considerable simulation experience.

To learn more about the influence of hands-on activities on students' conceptual understanding of statistics, this paper examines their use to introduce probability distributions, sampling distributions, and the Central Limit Theorem. In particular, the paper reviews sample classroom activities, examines students' understanding of the underlying concepts, and seeks to identify connections between students' conceptual understanding and their classroom experience.

Method

Subjects (N = 10) were students enrolled in a summer graduate course at Montana State University, *Statistics for Teachers*. Although it is a graduate course, *Statistics for Teachers* focuses on the conceptual underpinnings of introductory statistical topics, including visual representations of data, probability and sampling distributions, regression, and correlation. The objective of the course is to provide students, all of whom are practicing or pre-service teachers, with the conceptual foundations needed to teach statistics at the pre-college level.

Subjects' statistical backgrounds varied. All the students had taken statistics, usually one introductory course, as part of their teacher preparation. In fact, many of the practicing teachers had taught statistics, although their knowledge of the subject was admittedly algorithmic. Pre-tests administered in the course revealed that subjects' conceptual understanding of statistics was incomplete, absent, or altogether wrong. Class instruction purposely modeled the recommendations of the National Council of the Teachers of Mathematics (NCTM, 1989, 1991). Typically, students completed sequences of hands-on activities that focused on one or more statistical concept. By reflecting on their explorations, forming and testing conjectures, and discussing their discoveries with their peers, students actively built an understanding of statistics.

Two forms of data were collected. First, open-ended, paper-and-pencil tasks were administered following the completion of each sequence of instructional activities. For example, with regard to sampling distributions and the Central Limit Theorem, students responded to the following questions:

(1)What is a sampling distribution of the mean? In a short essay, outline what you know about sampling distributions of the mean.

(2)What is the statistical significance of the sampling distribution of the mean?

Follow-up interviews composed the second form of data collection. Specifically, if students exhibited misconceptions or their response reflected classroom experiences, follow-up interviews were conducted to provide additional insights about the response. In these interviews, students reviewed their responses and explained how they arrived at their solution. Additional questions probed students' thought processes and sought to identify the origins of their conceptions.

Sample Classroom Activities

Although hand-on activities were used throughout the course, this research focused on activities related to probability distributions, sampling distributions, and the Central Limit Theorem. As an illustration of the instruction that charcterized the course, consider the following efforts to introduce sampling distributions and the Central Limit Theorem. Initially, students drew 20 samples of size 4 (with replacement) from a jar containing numbered slips of paper and calculated the sample means. The data were then pooled and students built histograms of the results, described the shape of the resulting distribution, and used their data to draw inferences about the true mean of the population. Subsequently, the experiment was repeated with samples of size 16 and students compared the two sampling distributions.

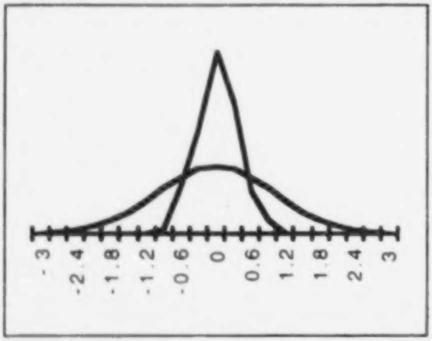

Figure 1. Sample TurboPascal program output.

At this point, the investigation was conducted via computer. A TurboPascal program constructed by the author allows students to select an underlying population, the sample size, and the number of samples to collect. The program then collects samples of the indicated size, computes the sample means, and graphs the underlying population distribution and the relative frequency polygon representing the distribution of sample means, as in Figure 1. In this way, students could compare the center and spread of the sampling distribution with that of the underlying population. Furthermore, since the population remained constant, students could vary the sample size and observe its effect on variability. In theory, an understanding of the Central Limit Theorem emerged as students observed the outcomes generated by large sample sizes and a large number of repetitions.

Results

Students' responses to open-ended questions reveal both correct and incorrect beliefs about the statistical concepts underlying each activity. For example, one question asked students to sketch two distribution curves with equal means but different variances. Four students provided correct responses and each of the remaining responses correctly represented the center of the populations. However, students' efforts to depict differing variabilities revealed fundamental misunderstandings of probability distributions. In particular, five students incorrectly nested one population within another, as in Figure 2.

Follow-up interviews revealed that the question was indeed problematic for students. For instance, the following comments are indicative of students' reasoning in this situation:

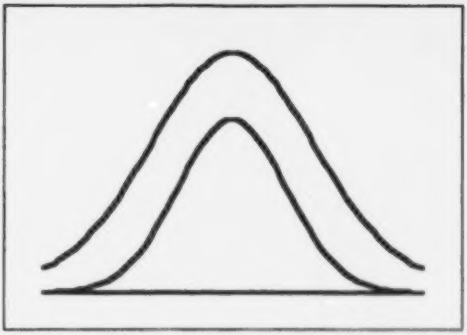

Figure 2. A nested representation of two distribution curves.

> "I kept asking myself, "How can I draw them so that the variances aren't equal." I knew that variance deals with data's (sic) spread, so what I decided to do was draw one population more spread out than the other."

What was more striking about the students' responses, however, was their inability to recognize the paradox of their representations. When asked to identify the area under each population curve, each student readily answered one. It wasn't until the impossibility of their representations was noted - that one population could not be nested within the other and each have an area of one - that students recognized their mistake.

With regard to sampling distributions, students also exhibited both understanding and misunderstanding. With regard to the latter, a number of students believed sampling distributions to be empirical, rather than theoretical, distributions of sample means. Additionally, several students referred to the importance of repeated sampling. This latter belief is of particular interest, for it reveals the influence of classroom experience on students' statistical thinking. For instance, KJ's response refers directly to classroom experimentation:

> "I learned that the sampling distribution of the mean is actually a distribution of sample means, that its center is the true population mean, and that its variance decreases as the sample size increases. Overall, the sample mean is good at estimating the true mean, especially if the size of the sample is large. What our experiments showed, though is that you've got to be careful. We can get good information with sample means, but only if we can take lots of samples."

Discussion

The data indicate that hands-on experiments can lead to conceptual understanding. For instance, although he lacked prior knowledge of sampling distributions, KJ correctly indicates that (1) sampling distributions of the mean are distributions of sample means, (2) sampling distributions are centered about the true population mean, and (3) the variance of sampling distributions is inversely related to sample size. On the other hand, the data also suggest that hands-on experiments can *contribute* to the formation of statistical misunderstandings. With regard to sampling distribution experiments, for example, several students mistook the process whereby sampling distributions were constructed - namely, the process of repeated sampling - for salient information about the distributions. Additionally, the use of experimental data to introduce sampling distributions persuaded students that they are empirical, rather than theoretical, distributions.

Although the influence is more difficult to discern, students' conceptions of probability distributions are also a product of their classroom experience. For one, students mistakenly believed that probability distributions are generated by (rather than the origins of) experimental data, a belief that was encouraged by the extensive use of hands-on experiments. Additionally, students' decision to nest one distribution curve within another seems to reflect their experience with histograms. Specifically, students conducted experiments, used histograms to represent the data, and identified qualitative aspects of their results. As more data were collected, students altered their histograms. Histograms composed of ten observations were nested within histograms reflecting 100 observations. Subsequent activity focused on relative frequency data. In seeking to understand distribution curves, however, students seemed to focus on their initial experiences. For instance, when asked if one distribution curve could be nested within another, student LS responded:

> "I think you can do that. I remember what, I used what we did first to build distributions. We started with some data and then graphed more data to get two distributions, and because of the more data, we got bigger distributions. One was inside the other. I guess that's why I think it's OK."

In theory, hands-on experiments promote the formation of rich conceptions of statistical ideas. As this research demonstrates, however, the resultant conceptions may only be partially correct. In particular, it appears that students attend to salient *and* non-salient features of the experiments, such as the repeated sampling of the sampling distribution activities and the progressive construction of histograms. In turn, the focus on non-salient features of in-class experiments contributes to the formation of incorrect or partially correct understandings.

Considering recent constructivist philosophy, these findings are not surprising. With our knowledge of statistics, we (statistics educators) observe the results of hands-on activities and readily extract the salient conceptual features. Students, on the other hand, seem to approach tasks holistically. Rather than extracting the salient features of an experiment, students draw their generalizations on the basis of *all* results and observations.

Without knowledge of the salient statistical concepts, they periodically draw incorrect or partially correct generalizations.

Conclusions

Hands-on activities are powerful educational tools. The results of this research, however, suggest that teachers must carefully monitor their use. Certainly, more research is needed to identify the benefits and drawbacks of hands-on classroom experiences.

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RESEARCH ON THE TEACHING AND LEARNING OF PROBABILITY AND STATISTICS

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The growing use of statistical reasoning as a qualitative tool and the recommendations of the current reform movement in mathematics education increase the importance of creating a content strand and the development of effective learning experiences in statistics and probability throughout the K-12 curriculum. Additionally, there is growing interest among university faculty in the reform of introductory, college-level courses. As curriculum developers, teachers, and teacher educators respond to this call, there needs to be an accompanying commitment from the research community to provide support for the reform effort through inquiry into issues related to the teaching and learning of statistics and probability at all age levels.

The purpose of this discussion group is to facilitate communication among individuals currently conducting research in this area and to spark the interest of others into learning more about the field. The session will begin with time allowed for participants to present brief summaries of current research. Discussion will then focus on identifying key issues that are important for research to address within the current reform efforts. A goal of the group will be to facilitate post-conference communications among participants. Issues that may be addressed include the following questions related to the area of statistics and probability.

- Students' beliefs and intuitive understandings
- Development of conceptual understandings by grade level
- Teachers' understandings
- The use of hands-on activities and technology
- The role of research

PRESERVICE ELEMENTARY TEACHERS' PERCEPTIONS OF SYNTACTIC AND SEMANTIC ELEMENTS IN BAR CHARTS AND FREQUENCY DISTRIBUTIONS

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"Graphicacy" or the ability to make sense of and use graphs is a critical aspect of statistical thinking, which depends, in part, on using a graph's syntactical elements to understand its semantic structure (Bright, Curcio, and Friel, 1996). This report uses the framework of "graphicacy" to examine preservice elementary education students' interpretations of bar charts and frequency distributions.

Through a classroom activity the students first predicted, and then recorded the number of water drops that they were able to fit on the head of a penny. As a homework assignment, the students drew a double bar graph of the predicted and actual values for the 16 measurements taken in class. The results of this assignment were unexpected, with students' work indicating deficiencies in their understanding of statistical graphs. To understand the students' thinking, a follow-up activity was constructed from the student-drawn graphs consisting of two representative bar charts of ungrouped data and two representative frequency distributions. Students were asked to assign a grade from 1 to 5 to each of these four graphs and briefly explain their grading choices.

An analysis of the follow-up activity found that, overall, students preferred the bar charts over the frequency distributions. The written responses show that many students had difficulty interpreting the syntactical elements of axes labels and scale markings in the two graph types, being unable to distinguish between the nominal scale used as the x-axis in the bar charts and the ratio scale used in the frequency distributions. These students could not assign the correct semantic meaning to the bars of the frequency distributions, and hence, make sense of that particular graphical representation. Their responses also indicated a lack of understanding of the statistical concept of frequency distribution. These findings help to illustrate the complex nature of the graphical and statistical understandings that are required to make sense of bar charts and frequency distributions.

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PROBLEM SOLVING

DEVELOPING PROBLEM-SOLVING CONCEPTIONS OF MATHEMATICS: A PRESERVICE TEACHER'S EXPERIENCES

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The nature and evolution of one preservice secondary teacher's beliefs about mathematics are described. The participant was interviewed and observed throughout the final year of her preservice program (methods course and student teaching). She communicated narrow (i.e, Instrumentalist (Ernest, 1991)) views about the nature of mathematics and her core beliefs changed very little during the study. The main evolution during the methods course was an increased awareness of reform themes. During student teaching, she refined some of her views but the most significant change involved her ability to elaborate her beliefs with specific classroom examples. Additionally, the paper explores relationships between her espoused beliefs and teaching decisions in the context of a reformed classroom environment.

This paper describes how a preservice secondary teacher's conceptions about mathematics, teaching, and learning evolved during a secondary methods course and student teaching. One characteristic of recent reform efforts in mathematics education has been an attempt to move teachers away from the sole reliance on formal lectures toward practices that support active and vocal participation by students in the learning process (NCTM, 1989). Students need opportunities to explore and communicate, and make sense of mathematical concepts and problems. Teachers, like their students, are thoughtful and reflective individuals who construct ideas about mathematics, their students' learning, and what it means to teach effectively (Cooney, 1994). Recognition of this notion has led to research focused not just on what teachers do, but also on what they think.

A different philosophy of mathematics is also reflected in the goals of reform. Mathematics is characterized by the creative and changing nature of its processes and ideas, and connections among them, not just by accurate results and infallible procedures. Teachers' views of mathematics are related to their beliefs about the teaching and learning of mathematics, and therefore play a significant role in shaping patterns of instructional behavior (Ernest, 1989; Lerman, 1990; Thompson, 1992). As such, attempting to better understand preservice teachers' beliefs about the nature of mathematics and examining their ability to change these beliefs have become important areas of study for mathematics teacher education. This paper is a description of one preservice teacher's beliefs about the nature of mathematics. Additionally, we provide insight into how her conceptions about mathematics are related to her enacted beliefs. We do so over an extended period of time (an entire academic year), enabling us to identify how her conceptions evolved over this period. These relationships are particularly relevant because they took place within the context of a preservice program aligned with the mathematics education reform movement described above.

Ernest (1989, 1991) defines three categories (Problem-Solving, Platonist and Instrumentalist) to describe beliefs individuals have about the nature of mathematics, and elaborates how these views provide a basis for teachers' conceptions of mathematics teaching and learning. According to the *Problem-Solving* view, mathematics is a continually growing field of human creation that develops through conjectures, the generation of patterns, proofs and questioning. It is not a finished product, but rather its results remain open to revision. Mathematics is seen as a "dynamically organized structure located in a social and cultural context" (1989, p. 250). Teachers with Problem-Solving views of mathematics are likely to think of themselves as facilitators who encourage students to pose and solve problems. Such teachers are inclined to value students' construction of their own understanding. The Platonist view suggests that mathematics is "a static but unified body of certain knowledge. . . . It is discovered, not created." This view involves a global understanding of mathematics as a consistent, connected and objective structure." A teacher with Platonist views of mathematics is likely to see his or her primary role as explainer. In general, a Platonist teacher emphasizes conceptual understanding of unified knowledge, and sees the students as "receptors of knowledge (1989, p. 250-251)." The Instrumentalist view maintains that mathematics is "an accumulation of facts, rules and skills to be used in the pursuance of some external end." Instrumentalism involves knowledge of "mathematical facts, rules and methods as separate entities" (p.250). The Instrumentalist view of mathematics is likely to be associated with an "instructor model of teaching," where teachers emphasize mastery of skills, rules and procedures and usually strictly follow a text or scheme. The student's role is to master what the teacher is telling him or her.

Often there is a disparity between espoused beliefs and what a teacher actually does in the classroom (Ernest, 1989; Cooney, 1985). In addition to views about the nature of mathematics, social context and the teacher's level of consciousness of her own beliefs can influence instructional practice (Cooney, 1994). Therefore, we believe that classroom context opportunity for reflection play significant roles in preservice teachers' development of Problem-Solving conceptions. This view not only influenced the design and implementation of the teacher education program in which Ashika participated, it also influenced the way we planned and carried out the study reported in this paper.

Design

Participant and Site

Ashika completed her undergraduate degree in mathematics at a private African American college, and during the period of this study, was working on a masters degree in education with secondary certification in mathematics (at a large midwestern university). She considers herself to have a strong mathematics background having been a successful student of traditional mathematics.

During the fall of 1995, Ashika was a student in a mathematics teaching methods course required for secondary mathematics certification. She participated in mathematics methods instruction that required her to explicitly connect her views, experiences, and understandings of reform themes. For example, with another student, she completed a multimedia project communicating her understanding of alternative assessment. This multimedia project also required her to reflect upon and critically analyze the practice teaching experiences that were occurring at the same time as the methods course. The second author was course instructor.

During winter 1996, Ashika student taught in a reform-oriented classroom at a public high school. Her cooperating teacher, Ms. Stevenson, had integrated cooperative learning, technology and large group projects into her curriculum for years. Each week Ashika also attended an informal seminar with other student teachers. During these seminars, discussions centered on reform themes and student teachers' reflections about their classroom experiences. The first author supervised Ashika's student teaching experiences and led the seminar.

Data Collection

We used an ethnographic case study design (Stake, 1995) to collect information about Ashika. Data were collected between September 1995 and June 1996. We engaged Ashika in formal and informal interviews, observed her student teaching, and collected written artifacts, including surveys and assignments from the methods course and student teaching seminar. Four semi-structured interviews (approximately one hour in length) allowed Ashika to elaborate survey responses and communicate her views about mathematics, the methods course, and her practice teaching experiences. During these elaboration interviews much of our discussion arose from the participants' comments. All formal interviews were audio taped and transcribed. Selected written assignments completed in the methods course and student teaching seminar were also analyzed. During student teaching, seven classroom observations occurred. Informal interviews (length varied from 20 minutes to one hour) followed these seven observations. Finally, Ashika was observed during 12 student teaching seminars.

Results And Conclusions

This section shows how Ashika's largely Instrumentalist views about mathematics did not appear to change much, despite experience with Problem-Solving themes during her preservice program. Additionally, it addresses the issue of how her beliefs were related to her observed mathematics teaching.

Like many other preservice mathematics teachers (e.g., Owens, 1987; Wilson, 1994), at the beginning of the methods course most of Ashika's communications were consistent with an Instrumentalist view of mathematics. She saw mathematics as being sequential, building from a foundation of arithmetic and algebra. In order to understand the more advanced areas of mathematics, Ashika claimed that it was essential to know the basics and how to apply them. As she explained during our first interview:

> *What you learned in elementary school is basic. I mean, you get this complex calculus problem and it's all going to boil down to you adding, subtracting. So if you get stuck at the foundation, then you can't move on....I don't see math as being very complicated...the concepts might be complex, but once you muddle through it and get to it, all I'm doing is adding. But, most people don't see that.*

Similarly, on several occasions Ashika referred to having to practice and memorize procedures in order to learn new material. According to Ashika, formulas are "a step-by-step kind of thing to come to an answer . . . if you have this formula, then you're sure, you know what you're doing, you're sure to come up with the right answer." To Ashika, doing mathematics mainly involved following a set of rules or formulas that will always lead you to the right answer.

However, Ashika did not see mathematics as consisting exclusively of these formulas. During her experiences in high school geometry and as a student of college mathematics, she discovered what she labels the "theory part," or proofs, of mathematics. She described why these experiences were unsettling to her:

> *Because, when you're younger, you think "Oh math is adding" . . . and you think that it's some application or some process or whatever, and then . . . . It happened to me twice. It happened to me in geometry and then it happened to me when I got into my first math theory class, that you think, "OK, I can do calculus or whatever," and then someone goes, "Well prove this." And then, and they're telling you, "This is the true math. That's just baby stuff, and anybody can do that. But, if you can do this, then you are really a mathematician." . . . . That's why it shocked me because I could do one, but I wasn't very. I had to work hard at doing the other. . . . Most people, they don't know that side of math . . . They think of math as being all formulas. . . . That's all you do is number crunching.*

Although she acknowledged that there was a non-"number-crunching" aspect of mathematics, she saw the theory part as being utilized mainly by mathematicians, and not understood or used by most people. While as a student she had to develop and understand proofs, she did not think that it was important for her students to understand this part of mathematics.

Like other students, Ashika claimed that her conceptions had not changed very much during the methods course. While we detected some change, the nature of that change was related more to an increased recognition of and familiarity with issues that she had not fully articulated before. For example, at the end of the course Ashika reiterated her beliefs that building a strong foundation is important at the high school level, and she continued to see arithmetic as being an essential "skill that you need no matter what you do." However, she also expressed that the basics need to become automatic so that students can focus on processes, concepts, and applications. According to Ashika, students need to master the basics before they are "ready to do the conceptual stuff, because they won't have to think about the manipulation stuff." As she stated, "How can you understand how to apply something if you don't understand what it is?" While this comment illustrates that Ashika still held the core belief that mathematics is mainly an accumulation of rules and skills to be mastered (an Instrumentalist view), her reference to applications, problems and concepts suggests that, at least to a small extent, she saw mathematics as being connected (Platonist).

Her movement toward a Platonist view of mathematics is further illustrated by one of her survey responses (and subsequent elaboration). Ashika discussed why mathematics is useful (to describe natural phenomena, explain why something happened, find answers to problems, and to predict future events). She spoke less of finding answers and using formulas, and more of students "thinking about what they are doing," and analyzing the process (i.e., the steps that one uses to solve problems). She began to articulate a belief that mathematics is an activity that enables students to think critically.

Throughout her student teaching experience, Ashika's views of mathematics influenced her practice and facilitated her reconciliation of some of her beliefs about the teaching and learning of mathematics. For example, at the end of her student teaching, Ashika talked about a lesson that she taught during a logic unit in geometry. She asked her students to make *Kool-Aid* from a recipe and discuss their problem solving process. She particularly enjoyed this lesson because she felt it "broke up the monotony, . . . involved something concrete that everyone could relate to," and helped the students become engaged and make "connections." When explaining the connections she hoped students would make, she discussed how she tried to convey to the students that to be efficient when following a recipe, you need to have all the ingredients ready in advance. Ashika then elaborated on this point and related her lesson to problem solving in general:

> *So, you need the skills to go to a problem, where you have all the ingredients. And like, an ingredient could be like just basic background information, formulas, and things like that, that all help in the problem solving process.*

Even though this comment occurred during a discussion about problem solving, the statement reveals her underlying belief that the foundation or essential aspect of mathematics is its formulas. Similarly, in the student teaching seminar she described how her students needed to be shown how to do two-column and paragraph proofs before they could successfully do proofs themselves. In this example, the foundation consisted not of procedures, but of the skills that were necessary for students to be able to focus on the idea and process of proof.

Throughout her student teaching, we observed Ashika reconciling her core conceptions of mathematics (e.g. mathematics builds from a foundation, which students must first master before they learn concepts) with her beliefs about how to teach and how students learn. For example, she struggled with the issue of how much she should tell her students, and when it was appropriate to let them explore: "What should I do if they are not getting it? How much to tell? What else can I do?" This is an example of a theme she did not mention very much during the methods course (although it was discussed at considerable length in the course). At the end of student teaching, Ashika explained what she considered her role as teacher to be.

> *In some form or another, to transfer knowledge, or to help students gain knowledge. It depends on exactly what you are doing. Some things students just have to be told, and then you move from there. But, there are certain things that you can explore together and that they can be part of, like the uncovering of the material and concepts, and things like that.*

Here we see a recognition that concepts are not only important, but that students should come to understand those concepts through exploration. It is not clear whether Ashika'a recognition of the importance and value of exploration occurred because she taught in a classroom where student exploration occurred on a regular basis, but these circumstances allowed her to connect this realization to actual teaching experience. We emphasize however, that Ashika's acknowledgment of the value of student exploration did not signify a shift in her core beliefs about the nature of mathematics. She did not even recognize that her core beliefs conflicted with Ms. Stevenson's. Ashika attributed most of Ms. Stevenson's innovative methods and the circumstances (e.g., teaching with no textbook) not to a difference in philosophy, but to Ms. Stevenson's experience.

Discussion

Ashika's case illustrates how impervious preservice teachers' beliefs can be to change. Despite being inundated with reform themes during her preservice experience, Ashika maintained largely Instrumentalist views of mathematics. Ashika claimed that during her student teaching she lectured very little. Although our observations verified this fact, most of Ashika's comments suggest that she will do more lecturing when she becomes a teacher. Ashika generally followed Ms. Stevenson's mostly innovative organizational and instructional formats. This may account for the discrepancies we observed. This raises two questions: (1) Is it necessary for teachers to hold Problem-Solving views of mathematics to teach in innovative ways?, and (2) Does experience with innovation during practice teaching guarantee that preservice teachers will adopt Problem-Solving conceptions? Evidence from this study suggests that the answer to both questions is no.

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THREE PERSPECTIVES ON PROBLEM-BASED MATHEMATICS LEARNING

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The complexity of classroom interactions offers challenges to researchers interested in students' thinking. One method of addressing this complexity is through multiple perspectives. We offer three portraits of a high school mathematics class in which a problem based, discovery oriented approach to learning was being introduced. These portraits describe students' reasoning, their perceptions of themselves as problem solvers, and the teacher's perspectives on changes in her students and herself.

Luhmann (1989. p. 11) reminds us that no system can completely understand its environment. For a teacher or researcher trying to come to grips with the complexity of a room full of students learning mathematics, the reminder is hardly needed. What is not being seen is always so much larger than what can be seen. Added to this are the constraints imposed by the same senses and patterns of thought which allow us to make sense of events in a classroom in the first place. We see things in terms of the perspectives and theories we bring to the scene (Maturana, 1988).

Recently researchers involved in the Classroom Reasoning Project have adapted the bricological research methodology of the Enactivist Research Group (Kieren, Gordon-Calvert, Reid & Simmt, 1995) to look at the development of students' reasoning in a grade 10 math class. Central to this methodology is Maturana's observation that "Everything said is said by an observer" (1987). By this Maturana means to make it clear that we cannot adopt a position from which we can objectively report what is going on. At the same time, however, we have the opportunity as observers in a social realm to observe the observations of others, and for them to observe ours. Through conversation we can begin to see the multiple aspects of a classroom (or another environment). We can never come to a single vision, in fact we would not want to, but we can create richer portraits of mathematical activity that would be possible from a single perspective.

In the following we will describe the mathematical activity of a grade 10 class from three perspectives. One of these perspectives is that of the classroom teacher herself, whose portrait includes her own feelings about her students' mathematical activity, as well as the perspective of an experienced teacher on effects of the research taking place in her classroom. A second perspective is that of a researcher concerned with the development of reasoning, whose portrait considers the teacher, the materials used, and the students activities in terms of the mathematical reasoning they developed. The third perspective is that of a researcher interested in group problem solving in the teaching of mathematics, whose focus was much more on the work of five students in particular, and how the introduction of problem solving and group work to their math class affected their attitudes towards themselves as mathematical problem solvers.

Background

Each of the following portraits is based on observations of students in a grade 10 mathematics class. These students had learned mathematics in a traditional way throughout their school careers, and for the first half of their grade 10 year. At the beginning of a unit on coordinate geometry, the students were assigned to groups, and given exploratory activities designed to occasion the learning of concepts central to the unit: the relationship between linear equations and graphs of lines, slope, and intercepts. During a typical class the students would begin working on a prompt, such as:

> "For each pair of parallel lines, find the equations of the lines, and say what is the same about the equations and what is different. Explain why."
> [Four graphs showing pairs of parallel lines were given.]

The students were expected to discover ways of finding equations of lines, and the significance of the slope and y-intercept through their explorations. At the conclusion of explorations based on a given prompt, the students made presentations to the class, summarizing their group's findings.

Most of the students had not been very successful in mathematics in the past, and several were repeating the course. The rate of absenteeism in the course was high. Two students' were working in their second language, and many students had weak reading and writing skills.

Portrait 1: A Classroom teacher's perspective — Glenda G. Cluett

At the onset of this project I felt much like a novice teacher; I had no idea of what to expect from my students or myself. My experience in planning lessons and anticipating troublespots seemed of little value here. I knew only that my students would explore and hopefully discover the intended curriculum objectives for linear sentences.

The experience was enlightening and unnerving. To change roles from purveyor of knowledge to facilitator is no easy task. Old habits do indeed die hard! It was stressful to watch students go awry in their problem-solving attempts and not to intervene. Resisting the urge to move things along at my normal pace was very difficult, as I was constantly aware of the time constraints in "covering the curriculum". I struggled to respond to students' questions with more questions that would spark further investigation, rather than to respond with answers. A deeply-ingrained philosophy of what it means "to teach" was being contradicted. Without conscious effort at all times to allow students to discover for themselves, I quickly reverted back to my old ways.

Initially I was sceptical of the possibility for success with my students. I was leery of their grasp of the necessary prerequisites to the topic, and equally doubtful of their reasoning capabilities. My students, I learned, were less confident in their mathematical ability than I was. It was exhilarating to watch weak, normally unenthused students get excited as they met with success, urging me to give them the next activity. Time on task increased substantially, so much so that I feared students would lose interest as they became fatigued. Gradually, student questioning shifted focus from me, the teacher, to other students. Students' overall difficulty with verbalizing their conclusions was emphatic of their belief that "doing math" is more important than "talking about it".

For both teacher and student a great deal more took place than merely a change of approach to learning, the value of which should not be underscored by the difficulties encountered.

Portrait 2: Reasoning in mathematics class — David A. Reid

In past research (Reid, 1995; Kieren, Pirie & Reid, 1994) I began to explore students' reasoning from an enactivist perspective. My current work represents an attempt to extend the insights I gained working with pairs of students to groups working under normal classroom conditions. One of my major concerns is the relationships between needs to explore, verify and explain and deductive and inductive reasoning. Here I will report specifically on students' use of inductive reasoning to explore and deductive reasoning to explain, and on their use of procedural explanations over structural explanations.

All of the students' discoveries, which included the relationship between the parameters m and b in $y=mx+b$ to linear graphs and the equality of slopes of parallel lines, were made inductively. The students examined several cases, saw patterns, and generalized. In some cases generalizations were suggested by other students and then confirmed inductively. This use of inductive reasoning to explore conforms in part with the model of mathematics set out by the NCTM (1989, p. 143) of a process of inductive discovery followed by deductive verification. The structure of the activities we used emphasised discovery by induction, in contradiction to one of my intents which was to see if students could explore deductively. A significant constraint which led to our structuring the activities in the way we did was the curriculum objectives we intended to achieve, which were oriented toward students knowing specific facts and using specific procedures. Knowing *that* became the main focus of our teaching, and induction usually provided the simplest path to that goal.

The students did use deductive reasoning, however, both in explaining things to their peers and when asked to explain on tests. These explanations were usually only partially formulated and involved very few steps, but their use confirmed that some students do use deduction in some contexts. An unexpected alternative to deduction emerged as well. In many cases students responded to requests that they explain why by describing the outcome of a procedure. For instance to explain why the expression $3x-2$ describes a given graph some students would write that when you make a table of values for the equation and the graph you get the same thing. It is not clear that the students were giving more than a procedure to determine that the expression describes the graph. In hindsight this is an obvious result of their developing understanding of linear equations, which (in the language of Sfard, 1991) was more operational than structural.

Portrait 3: Perspectives of problem solving — Debbie M. Blackmore

I focused on one small group within the large class setting. Of particular interest were the interactions among group members during the problem solving sessions, as well as the perceptions held by these students concerning their problem solving abilities. The members of the focus group were interviewed and observed subsequently as they solved several process problems, both individually and cooperatively.

Results of my investigation suggest that within any given group of students there may exist factors over which teachers have little (or no) control. Language differences and chronic absenteeism are examples of such factors that became apparent during this study. These may require special consideration in situations in which students rely on each other, as in a cooperative learning environment. In the focus group only one of the five students was present for all thirteen classes while two students missed more than half the classes. The sporadic presence of these students appeared to affect the group's ability to work cohesively. Also, for one of the students in the group, language was a barrier that may have had some influence over the group's ability to communicate effectively.

Discourse among group members may also have been a factor contributing to the performance level of the group. In fact, students in this group tended to work more as independent students who sat together and compared answers than as a consolidated group who worked together to solve a problem. However, it is noted that "limited verbalizations do not preclude the occurrence of meaningful and influential interactions" (Hardy, 1995). I was also observed that these students' appeared to be reluctant to share and discuss their mathematical ideas. Interviews revealed that these students lack self-confidence and perceive themselves as "weak students" with little to offer their colleagues. The interviews also revealed that the students indicated a preference for group work in the mathematics class, although it was primarily for purposes of "getting the answer". This suggests that it may not be enough for teachers to simply group students. It is imperative that teachers be cognizant of the fact that students must be assisted in learning *how* to work cooperatively, with a focus on solving problems, not merely arriving at an answer. Long term effort would be required to facilitate student development of both cognitive and metacognitive skills necessary for successful problem solving. It is not the intent of this study to imply that accomplishing this is an easy task but one worthy of the endeavour.

Conclusions

Providing three perspectives on students' learning of mathematics in a single classroom barely touches on the complexity of the situation, and the limitations of human communications attenuate the richness of our three perspectives even further. For a reader as an observer of our observations there is an opportunity to make a different sort of sense of what we offer than we can ourselves, in a way providing still another perspective. We cannot hope to convey what we have seen, which is less than what was there, but we can hope to have shown three ways of seeing what happens in a classroom, perhaps offering possibilities for teachers and researchers engaged with students' learning.

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RECONSTRUCTION TEACHERS' KNOWLEDGE OF INTERVENTION IN TEACHING PROBLEM SOLVING

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This paper reports on a study[1] involving an inservice program framed in narrative inquiry and a constructivist perspective of learning and its effect on teachers' thinking and teaching of mathematical problem solving, focusing on teacher intervention. The participants were six elementary teachers. Data collection included interviews and classroom observations. Data analysis involved determining the nature of the knowledge the teachers constructed and of changes in their teaching. The results indicated that the teachers constructed knowledge and made changes to their teaching in a way that was consistent with meaningful intervention, thus suggesting that the inservice approach used is a promising way to facilitate change.

In recent years, there has been growing acceptance that the teacher is the ultimate key to educational change. This suggests that teachers must change from their traditional approaches to teaching to better prepare students to live in a rapidly changing post modern world. However, how to accomplish this change is still questionable. Inservice programs for teachers have traditionally been prescriptive, an approach that is now viewed as problematic in facilitating meaningful outcomes for the teacher when fundamental changes to his or her teaching are intended. Alternative approaches that are now being advocated have shifted to a more humanistic perspective in which teacher development is considered in terms of self-understanding with a focus on teacher reflection and teacher biography. The argument here is that self-understanding comes before meaningful and substantial changes in teacher behaviour. In mathematics education, in response to the need for teacher change studies involving the mathematics teacher's knowledge, beliefs, practices and learning have begun to gain prominence (Cobb, Wood & Yackel, 1990, 1991; Cooney, 1985; Hoyles, 1992; Ponte, 1994; Simon & Schifter, 1991). In particular the constructivist perspective has emerged as a promising route to more effective mathematics teacher development in some of these works. However, a focus on self-understanding still seems to be under-represented in studies of mathematics teachers. This paper is intended to contribute to the self-understanding, constructivist path of facilitating change in mathematics teachers' practice focusing on problem solving instruction.

The teaching of mathematical problem solving can be a challenging task for teachers because of the complex nature of genuine problem solving. Charles, Lester and O'Daffer (1987) described problem solving as "the coordination of knowledge, previous experience, intuition, attitude, beliefs, and various abilities" (p. 7). Consistent with this, as noted by Lester (1985),

> The primary purpose of mathematical problem solving instruction is not to equip students with a collection of skills and processes, but rather to enable them to think for themselves (p. 66).

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To accomplish this purpose, the teacher's behaviour, in terms of intervention in the students' processes when teaching problem solving, becomes a critical consideration. Teacher intervention should facilitate rather than hinder the development of the students' ability to think for themselves. This paper reports on an inservice program and its effects in facilitating teacher development of this type of intervention. It is based in a broader study to develop an inservice program for teachers with poor backgrounds in mathematics and to investigate the effects of this program on the teachers' thinking, attitude and teaching of mathematical problem solving.

Theoretical Framework of Inservice Program: The inservice program was framed in the constructivist perspective of learning and the humanistic perspective of teacher development, mentioned earlier, focusing on narrative inquiry. The constructivist perspective recognizes that knowledge cannot be acquired passively and can only take place when existing cognitive structures meet with perturbations. von Glasersfeld (1989) emphasized that the most frequent source of perturbations is the interaction with others. Placed in this context, mathematics teachers' learning can be viewed as both as individual and an interactive activity, during which the teachers construct their own meaningful knowledge by engaging in and reflecting on experiences with mathematics, and negotiating personal meaning of these experiences with peers.

With respect to narrative, although it has a long history of being an important way of creating meaning, it is only recently that it has been advocated and adopted by researchers as a formal way of making sense of human actions (Bruner, 1990; Sarbin, 1992; Polkinghorne, 1988). This shift is associated with the view that narrative is the primary form by which human experience is made meaningful.

> Narrative is a meaning structure that organizes events and human actions into a whole, thereby attributing significance to individual actions and events according to their effect on the whole.
> [Polkinghorne, 1988, p. 18].

Thus narratives provide valuable means of understanding the past events of one's life and for planning future actions or facilitating change.

Based on these two perspectives of knowing, the key ideas that provided the framework for the program were:

I. Teachers should understand their personal practical knowledge to better facilitate meaningful and substantial changes in their teaching.

II. Teachers should critically examine, through their personal experiences, the nature of problems and the process of solving problems as a basis for understanding how to teach problem solving.

III. Teachers should be placed in a learner's role to experience genuine problem solving in a setting which fosters individual and social construction of problem solving concepts.

IV Teachers should reflect on both the affective and cognitive aspects of their problem solving behaviours to understand problem solving in a more realistic and humanistic way.

The Inservice Program: The program consisted of the following three stages.

Stage 1: Understanding self

Using the narrative approach, the teachers reflected and resonated in stories of personal experiences that involved, for example, their feelings and beliefs about problem solving, solving real world problems, and teaching problem solving.

Stage 2: Intervention to create disequilibrium

This stage involved several activities centered around the teachers solving a variety of non-routine mathematical problems working individually and in groups of two and three. Each activity was followed by group reflections on the experience, focusing on their feelings and thought processes. The researcher functioned as facilitator in stimulating reflection through the use of open-ended questions.

The activities intended to facilitate a new understanding of intervention involved the teachers working in pairs and groups of three, taking turns being the "teacher" and "student(s)" while solving problems. This involved the following scenarios:

Teacher passively observed an individual student.
Teacher passively observed a group of two.
Teacher actively intervened working with an individual student.
Teacher actively intervened working with a group of two.

In these roles the participants were given specific questions to guide their reflections. In the role of student, for example, they were required to consider, (i) when and why they needed "help", (ii) how they wanted the "help" given. In the role of teacher - passive observer, they considered, for example, when, how and why they wanted to give help and in the role of the active teacher, when, how and why they gave help. They then reflected on connections to their teaching of problem solving.

Stage 3: Understanding shifts in self

Using the narrative approach the teachers reflected and resonated in stories of their experiences in stage 2 in relation to their conceptions of intervention in stage 1 and their intentions regarding their future behaviour in teaching problem solving.

Research Process: The participants were six elementary teachers (grades 3 to 6) who were interested in improving their teaching of problem solving. They participated in the inservice program over a four-week period during their summer break.

All oral aspects of the inservice program (eg. group discussions and reflections) were audio-taped and all tapes were transcribed. Copies of all written work were also obtained. The teachers were observed in their classrooms while teaching problem solving, before and after their participation in the inservice program to document non-verbal behaviours and classroom organization that seemed to be integral aspects of their teaching. The teacher-student verbal interactions during these lessons were audio-taped and transcribed. Each observation was followed by an open-ended interview on the participants' thinking and feelings about what they did. All interviews were audio-taped and transcribed.

Data analysis involved determining the nature of the knowledge the teachers constructed and the effect of the inservice program on their teaching. For the former the transcripts of the experience were examined to determine the group beliefs reflected in stages 1 and 3 of the program in terms of how they conceptualized problems, problem solving, goals of teaching problem solving, teachers' role in teaching problem solving, and nature of intervention. These findings were shared with the teachers to get their feedback and corroboration of the accuracy of them. The effect of the program was determined by comparing data of their teaching before and after the program; comparing these differences to the teachers' description of their teaching after the program; getting the teachers' feedback on theses differences and using the transcripts to resolve and negotiate discrepancies. Comparison of their teaching was guided by a list of factors drawn from the NCTM standards (1991), other sources on teaching problem solving (eg. Charles & Lester, 1982), and the knowledge the teachers constructed, which was used to code the data. This list consisted of factors like: using non-routine problems and cooperative learning groups; allowing and exploring alternative strategies and solutions; recognizing students' thinking; asking non-leading questions.

Results and Discussion: The teachers individually and collectively constructed and recovered knowledge they were not previously aware of regarding themselves as problem solvers, problem solving and their teaching of problem solving. However, only the results relevant to intervention will be presented here focusing on the knowledge the teachers constructed or recovered and their classroom behaviours.

Knowledge constructed/recovered: The process allowed the teachers to recognize and understand misconceptions in their beliefs about intervention when teaching problem solving and to construct alternative conceptions in ways that were more meaningful to them as teachers. For example, prior to the inservice experience, each teacher believed that the only purpose of intervention was to help students to get her answer by her method, the only one possible. As one teacher explained:

> I always, um, before what I did, I looked at the answer. ... I looked at the written part that was important to me. ... I guess I really considered my way was the only way. ... I never looked at the justification before. It was just like, it didn't fit where I was intending it to be. Or I would say, "Well, that is not what I was looking for." [grade 3 teacher]

By the end of the inservice experience, collectively, they had constructed the following views about intervention:

(1) *In general, intervention should not be to tell students how to get the answer, but to stimulate their thinking to get over barriers and to make sense of their processes.*

(II) *Intervention can be passive or active.*

(III) *Active intervention should occur in 3 situations -- when students are stuck, off-track or lost, in the context of their approach or thinking.*

(IV) *The goal and nature of the intervention and who (teacher or student) initiated the intervention is dependent on which situation in (III) is involved.*

For example, students should initiate the intervention when they are stuck or lost. When the situation is "stuck", the teacher should intervene by asking three questions: Why are you stuck? What have you tried? What else can you do/try? If still necessary, this should be followed by an open-ended suggestion of something to try.

(V) *Intervention too soon or too late will likely trigger negative emotions and be counter productive for the students.*

Classroom behaviour: All of the participants made significant changes to their interventions when teaching problem solving. In general, there was a shift towards the way they conceptualized intervention as summarized above. However, since the teachers were left to determine how to change their teaching, each decided on what was meaningful and important for her particular situation and when and how to integrate the new knowledge into it. Thus each teacher's classroom process reflected different emphasis of the knowledge they constructed. The following excerpts from three teachers (grades 4, 3 and 6 respectively) provide a sample of this.

> This year I have really concentrated on not jumping in for them and saying, "Yes, that is right! That is right!" because I used to feel so happy that they were getting close to the answer. Now, instead, I will question them and say, "Well, what do you think?" "What about the looking back stage?" "What are going to do now?" So I try not to wrap it up quite so nicely for them.

> Well for me, it is focusing more on what students are thinking. ... And it really came home when I started going around and talking to kids. ... that for as many kids in the class there could be as many different alternatives or ways of getting to the answer. ... Because sometimes I still have the tendency to say, "No!" and then I stop and say, "Yes, ok, that was one way of looking at it" because this person was sort of off track, but then when they started explaining, I thought, "Ok, it is alright." You know it is justified.

> It is hard because you do have that answer in your head, and you do want them to come up with that answer and so you have got to let go of that and you got to let them make their own discoveries. So I keep trying to do this, to ask the kids questions in the right way and not forcing them to come up with my answer. ... It is not just me saying here is a problem and lets work through it and I am trying to get the kids to come up with the answer. It is a whole big thing now. ... I can talk more freely with the kids because my understanding is better and I have learned things that I never learned in school. ... This year I feel better that I can say.

All of the teachers pointed out that their teaching after the inservice program was more challenging, but more interesting and rewarding particularly because they were learning a lot from the students. They also reported that they had started to extend this teaching approach to all areas of mathematics because they all involved problem solving.

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Conclusion: Allowing the teachers to work form personal experiences to self-constructed theories of practice seems to be a meaningful way to facilitate change and deserves further consideration, particularly with respect to its long-term effects.

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TEACHERS AS PROBLEM SOLVERS/PROBLEM SOLVERS AS TEACHERS: TEACHERS' PRACTICE AND TEACHING OF MATHEMATICAL PROBLEM SOLVING

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This study investigated the relationship among three high school mathematics teachers definitions and beliefs about mathematical problem solving, their problem solving practices and how they teach mathematical problem solving. Each teacher was interviewed three times and observed once during a problem solving lesson. Data comprised of transcriptions of audio tapes, field notes, and completed problem solving checklists were used to prepare case studies. While the definitions, practices and teaching of the teachers varied, the findings were consistent within each case. The results suggest that how teachers are taught and what they learn as students is related to how they teach mathematical problem solving.

A commonly held theory is that teachers teach as they were taught (NCTM, 1991; MAA, 1991; Lortie, 1975). I believe not only how mathematics is taught influences teaching; it may be what mathematics is learned and practiced. Prospective teachers learn to do mathematics in many places; the mathematical habits they learn are part of the knowledge they bring to teaching. I think there may be a relationship between a teacher's own mathematical knowledge and his or her teaching of mathematics. The purpose of this study is to investigate this possible relationship. That is, is a teacher's knowledge and practice of mathematics related to how they teach? For the purpose of this study I considered mathematical problem solving, since it is a part of all mathematics.

The relationship I investigated is different from the theory suggested by Lortie (1975). Lortie was interested in the relationship between the teaching observed by teachers when they were students and how they teach. The theory I explored can be placed along side this. I am interested in determining if there is a relationship between *what* teachers learned as students of mathematical problem solving and their approaches to teaching. I explored what teachers learned, that is what they came to know and understand about mathematical problem solving and how they presented mathematical problem solving when teaching.

Method

In this study, I used qualitative methodologies to investigate a naturally occurring phenomenon (Patton, 1990). I wanted to investigate the relationship among what teachers believe to be mathematical problem solving, what they practice when solving problems, and how they teach their students to solve problems. This study has a multi-case design. I worked with three women who are respected high school mathematics teachers. The design of this study used techniques of analytic induction described by Bogdan and Biklen (1992).

I used three interviews. The first interview served to develop each teacher's definition of mathematical problem solving. At the second interview I asked her to solve three problems. Using the work she did as a starting point, we talked about her problem solving style and habits. After the second interview, I asked each teacher to invite me to observe a class where she was teaching problem solving (using her definitions). The third interview followed the observation; we discussed the lesson she presented and how she taught problem solving. All of the interviews and the teaching episode were audio-taped.

My data consisted of transcripts of the interviews and the problem solving lesson, my field notes, and the work done by the teachers when they solved three problems. Case studies were written after separating the data into three sets: what each teacher said about problem solving in general, what she did and talked about doing when solving mathematical problems, and what she did or said about teaching problem solving.

Case Studies

Karen

When Karen talked about problem solving, she included all types of problems encountered in mathematics classes, word problems and routine, drill-type problems. Problem solving, according to Karen, involves building and using a bag of tricks. That is, creating a cumulative collection of skills a problem solver has at her or his disposal and uses while solving problems. A successful problem solver, according to Karen, thoughtfully uses this bag of tricks to obtain efficient and accurate solutions. Word problems present a special challenge for problem solvers. Karen noted that the ability to translate from English to mathematical symbols is a skill problem solvers should have in their bag of tricks. She uses and refers to this skill when she was solving problems herself.

Karen tries to solve problems as efficiently as possible. First she reduced the problems to what she called the "bare facts." These facts were primarily the numerical data she obtained after reading the problem, the result of her translating the problem to mathematics. Next, Karen determined how the bare facts were related, using arithmetic or algebra, to get the problem to a purely mathematical form. She highlighted important details by drawing boxes around some of the numbers she obtained from the problem. Once the problem was rewritten in a mathematical form, Karen calls on the mathematical skills she had mastered to solve the problems; she uses her own bag of tricks.

According to Karen, her job when teaching problem solving was to add skills to the students' bags of tricks. She also wanted to teach them to use their skills efficiently. She mentioned that a problem solver's bag of tricks was cumulative, formed over time. When teaching, she connected the lesson to what came before, having her students build their own bags of tricks. Karen said she does not tell students how to solve a problem, but acknowledged that she does teach them steps needed in a variety of problem solving processes. In addition, she mentioned she tries to push students toward using efficient methods because using fewer steps will save them time and make them more likely to reach accurate conclusions. Karen's definition of problem solving is evident in her teaching of problem solving.

The work Karen did at the board while teaching problem solving was very deliberate. She often drew boxes or circles around key pieces of the problem. Karen also showed all the work on the board, doing computations beside the original problem for the class to see. This was reminiscent of her own problem solving work. She drew boxes around important information and laid out all the work she did to solve the problems herself. It seemed she was presenting her own style of problem solving to her class: identifying key information, showing work, and striving to be efficient in both contexts. She taught the problem solving she practiced.

Sara

When Sara talked about problem solving, she described the process as a form of experimenting. That is, gathering and analyzing data to make sense of the problem, then organizing the data to further explore the problem and reach a solution. Sara viewed problem solving as a creative activity not to be limited by rules or formulas. Sara stressed that most of what she teaches is not problem solving. She was required to cover a prescribed curriculum that involved teaching algorithms. When engaged in this type of mathematics, students do not have the opportunity to experiment, so Sara feels they are not engaged in problem solving. In order for a situation to truly represent mathematical problem solving, the problem must present a novel situation that cannot be solved immediately and may be solved by using a variety of methods. Throughout our discussion of problem solving, Sara referred to problem solving as an enjoyable undertaking; it should be fun for the problem solver. Part of the fun comes from experimenting and playing with the problem; some of the fun results from collaborating with others when working on a problem. Sara believes problems should be solved with others, not alone.

Sara enjoyed solving the problems I gave her. She laughed several times and took copies of all problems she did not do to solve later. Shortly after she read each problem, Sara began to make sense of the problem by quickly writing some notes to herself or by drawing a sketch. After this initial phase of jotting down information, she returned to the problem and organized the data, sometimes in her mind and other times on the sketches.

In general, Sara said she tries to get a problem into a visual form, a diagram or table of data, before beginning to figure out the actual solution. If she is unable to make sense of a problem using these strategies, she will use guess and check, not only to find the solution, but to make sense of the problem. Once the problem makes sense she can usually use algebra or calculus to solve it rather quickly. Sara preferred not to work alone when solving problems. She said she enjoyed, not only the company of friends when solving problems, but what she learned as a result of the collaboration.

Sara had her students work in groups for her problem solving lesson. When setting up the lesson, Sara presented a complicated problem to the class and told them what she expected them to turn in their solution (in the form of a written report) a week later. After briefly explaining the problem, Sara let the students work. She checked in on various groups and encouraged students to take time to play with the data and make sense of the problem before formulating a solution. As Sara interacted with the groups, she was playful, laughing and joking about the problem. She often encouraged students to talk to each other before she addressed their questions.

Sara's view of teaching mathematical problem solving reflected her definition of problem solving; an open, exploratory activity to be done with others. When teaching problem solving, she encouraged her students to doodle as a way to make sense of the problem; she did just that when solving problems herself. She drew pictures and wrote notes to herself. Sara acknowledged that her doodling was a little different from that of her students since she was already good at algebra but agreed that she did doodle. Sara said she learned when she had the opportunity to interact with friends while solving problems. She became one of the students she referred to when she talked about students learning when they discuss their solution methods. Sara, when teaching, gave students a chance to learn in the same way she felt she learned, with others. Sara taught the problem solving she practiced.

Kim

Kim considered problem solving to be like solving a puzzle. Solving a problem involves bringing any and all skills one has to the situation at hand. All of these skills, as long as they are logical, are valid to use when figuring out the puzzle presented in the problem. According to Kim, a situation is not problem solving when the freedom to use any strategy that makes sense to the problem solver is removed. That is, if a prescribed sequence of steps must be followed to reach the solution then the puzzle aspect of the problem is destroyed, and the activity is no longer problem solving. Problem solving, according to Kim, does not necessarily depend on the problem; it depends on how the problem is addressed by the solver. If given a certain degree of freedom and autonomy in determining how to reach a solution, many problems done in a mathematics class can be undertaken as a problem solving opportunity.

When Kim solved problems she liked to begin by making sense of the problem. To do this, she drew a picture whenever possible; she found problems easiest to understand if she could see what is happening. When solving problems that are not conveniently represented by diagrams, Kim said she may use the data in the problem to look for a pattern in order to find the solution. Once she understands the problem, Kim usually uses algebra as a tool to reach the solution. From the diagram or pattern she has uncovered she will write an equation to solve.

When she taught problem solving, Kim gave her students problems that could have been solved in many ways. She reminded students of possible alternatives when giving instructions, telling them they could "guess and check," "draw a picture," "make a chart," etc. Kim told her students they could use any method they wanted, as long as they could justify their answers logically. Students were assigned to groups for the day. The assistance Kim gave students was limited to defining terms found in the statement of the problem and critiquing their logic. She did not tell students what to do to solve the problem; she went along with their thinking when providing assistance. She taught problem solving consistent with her definition of the term.

As a problem solver, Kim used pictures and looked for a pattern in the given information to get started. She said, when talking about teaching problem solving, that she liked to see students draw pictures or handle manipulatives in order to visualize problems. Kim mentioned this same technique when she talked about how she liked to solve problems; she looked for ways to visualize the situation before using algebra to reach a solution. The problem solving Kim taught was reminiscent of the problem solving she practiced.

Conclusions and Recommendations

In my study I found there existed a relationship among what three teachers said when talking about mathematical problem solving, did when solving problems and taught as problem solving. I was careful to avoid looking for, or investigating, any causality among these three areas. It is worth further study to see if there is an order in changes teachers make when rethinking mathematical problem solving. That is, what comes first: a change in how they solve problems or how they teach problem solving? If such a cause and effect relationship existed, then teacher educators could use this knowledge in their work with both in-service and pre-service teachers.

As a mathematics teacher educator, in light of the current reform in mathematics education, I have felt compelled to change my teaching practices. I have done this based on intuitive notions that teachers need to be taught in ways they will be expected to teach. Part of this intuition resulted from thinking about Lortie's (1975) notions of teachers undergoing an apprenticeship-of-observation. So, in addition to undergoing this apprenticeship, I wanted my students to re-learn mathematics in ways congruent with the mathematics they would be expected to teach.

This study reveals a relationship among teacher's definition of problem solving, their problem solving practice and their teaching of problem solving. Wilson, Fernandez and Hadaway (1993) claim that since it is the teacher who creates the environment where mathematics is encountered and learned by students, the teacher must be the first one in the classroom to learn the mathematics. This claim is echoed by NCTM (1989, 1991), MAA (1991) and the intuitive notions I used in changing my teaching practices; teachers need to be prepared differently if they are to be able to teach the mathematics curriculum called for today. The results of my study validate this claim. The call for change in the pre-service and continuing education of mathematics teachers must not go unanswered.

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TRANSFER AND FORMULATION OF PROBLEMS IN THE LEARNING OF MATHEMATICS

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What instructional activities help students develop a mathematical disposition in which they can use their resources to solve problems in different contexts? This has become an important question related to the transfer of mathematical ideas. Although the way students participate during the classroom activities and the tasks used to promote such participation play an important role in the students' understanding of mathematics, there is a need to investigate what actually happens during the implementation of problem solving activities in the classroom. This study documents the importance of considering activities that include posing or reformulating of problems to promote transfer in the study of mathematics.

Introduction

The classical view of transfer gives emphasis on whether or not students who learned specific content in a situation can use it to solve problems that are different from the one studied during the initial learning. Greeno, Smith, and Moore (1993) discuss an alternative view which takes into account not only the attention to *affordances* or essentials related to the situation but also to the social interaction in which the learning activities take place. "In order to result in transfer, instruction should influence the activity so that it includes attention to affordances that are invariant across changes in the situation and that will support successful interactions in situations that have been transformed (p.104). In this context, it is essential to identify what are the invariants associated with the content or mathematical problems and promote learning activities that allow students to assimilate the essence of the task during the initial learning. An important activity that has been neglected in the study of mathematics is the process of formulating or redesigning problems from the students' perspective. The idea that this activity could help students analyze the main structure of problems and identify the invariant associated with those problems opens the possibility that the process of designing or completing problems could play an important role in the students development of transfer. In this perspective, the purpose of this study is to document to what extent the activity of reformulating or posing problems helps students transfer their knowledge to solve other problems.

Subjects, Type of Problems, and Procedures

Six pairs of students, all volunteers, were asked to work a series of problems at the end of their first calculus course (grade 12). They worked on the problems for about an hour (thinking aloud). There were two instructional activities with the entire group led by the researcher previous to the interviews (three hours each session). Here, the students were asked to work in small groups of four students on some problems and discuss specific questions related to the identification of the deep structures of the problems. It is important to mention that the instructor of this group frequently asked his students to work in small groups during the development of the class. That is, working cooperatively during class was not a new activity for the students. Activities in which students were asked to reflect on what aspects are essential to the situation or problems include working on a given problem and generalizing or analyzing some particular part or extension of the original statement. Types of problems used during the sessions included those in which the students were asked to formulate the question. The idea here was to explore the students' ability to perceive the important information needed to state the question. Two more variants of this type of problem were also given to the students, one in which some information was lacking and the students had to complete the information in order to formulate the question. The point here is that the missing information could be identified only when the deep structure of the problem was perceived by the students (for example, given two circles. The radius of one is 3 cm, and the distance between their centers is 10 cm. Do the circles intersect? (*We need to know the radius of the other circle*). The other type of problem includes those in which there was supplementary or useless information and the students were asked to select the minimum number of facts needed to solve the problem and explain why the other information was not necessary.

The purpose of the two sessions led by the researchers was to help students to identify essential components of the problems and to show that posing or formulating problems could be an important activity during the study of mathematics. Using Greeno's terminology, it was important for the students to be "attuned" to the situation. The students' interviews were carried out a week after the course had ended. Although the intervention of the interviewer was minimal, he was ready to provide help when required by the students. The problems used during the interviews were chosen to document to what extent the students showed mathematical arguments and strategies studied during the development of the instruction. Thus, it was important to ask students problems that did not have the same structure of those discussed previously during the class activities. The first problem in which the students had to propose a method of solution and to support it through a mathematical argument was:

> Nominations have been taken for mathematics teacher of the year. Five candidates have been nominated and an election has been conducted in which students voted by ranking their 1st choice, 2nd choice, 3rd choice, etc.. The results of the balloting by the 55 students are shown below:

Rankings	Number of Voters					
	18	12	10	9	4	2
1st	a	b	c	d	e	e
2nd	d	e	b	c	b	c
3rd	e	d	e	e	d	d
4th	c	c	d	b	c	b
5th	b	a	a	a	a	a

Who is it to be declared as winner?

The idea here was to document to what extent the students were able to utilize mathematical resources previously studied to support their responses and whether or not they could identify the main structure of the problem.

The second problem was chosen to be within the application range problems. Here, it was important to document the students representation of the situation and the resources shown to find the pattern associated with the generalized case.

> A six-lane track for running footraces is in the shape of a rectangle whose length is 1.5 times its width with a semicircle on each end. Each lane is to be 1 meter wide. What is the length and width of the rectangle if the inside track is to be 1500 meters long? For a 1500 meter race, the inside runner would start at the finish line. Where should the runners in the other five lanes start? What about if there are n lanes?

Frame of Analysis and Procedures

To analyze the information shown by the students, it was important to focus on to what extent the students dealt with aspects related to the deep structure of each problem during their solution processes (Schoenfeld, 1985). Each problem was discussed extensively previous to the interviews. The idea was to find several methods of solutions, to identify their deep structure, and to discuss possible connections or extensions to other situations. During the analysis, it was important to discuss the type of resources that the students showed to understand the statement of the problem and to work on some method of solution. There was special interest to document the extent to which students showed strategies and resources that were emphasized during instruction. For example, the use of a table or systematic list was important when the students worked the first problem. For the second problem the use of diagram and working with π instead using 3.1416 helped some students to see a useful pattern to approach the generalized case. These types of resources and strategies were important to analyze the students' work. It is important to mention that the students who participated in the study had taken a calculus course. The instructor of this course frequently asked his students to work on nonroutine problems in small groups. The students were used to discussing different approaches to the problems and often they had to defend and communicate their ideas to the class.

Results

While working the first problem, three pairs of students decided to assign some weights to each vote. For example, Peter and Anne gave five points to each vote given to the first place, four to each vote of the second place, three points to each vote given to the third place, two points to each vote given to the fourth place, and one point to each vote given to the firth place. All the students who assigned weights to the different places used the same numbers (1, 2, etc). Here, it was clear that these students had identified features related to the deep structure of the problem and their work was directed to deal with it. They used a new table to rearranged the initial information.

Teacher	Ranking					
	1st	2nd	3rd	4th	5th	
a	18	0	0	0	37	55
b	12	14	0	11	18	55
c	10	11	0	34	0	55
d	9	18	18	10	0	55
e	6	12	37	0	0	55

Using data from the table and the assigned weights, they calculated the following values:

$$a = 5(18) + 1(37) = 90 + 37 = 127; \quad b = 5(12) + 4(14) + 2(11) + 1(18) = 146;$$

$$c = 5(10) + 4(11) + 2(34) = 50 + 44 + 68 = 162; \quad d = 5(9) + 4(18) + 3(18) + 2(20) = 221$$

$$e = 5(6) + 4(12) + 3(37) = 30 + 48 + 111 = 189$$

a	b	c	**d**	e
127	146	162	**221**	189

With this information they identified ***d*** as the winner.

Other pairs of students went further than this; they decided to calculate the probability to get the first place for each candidate, that is P(a) = 18/55; P(b) = 12/55; P(c) = 10/55; P(d) = 9/55 and P(e) = 6/55. Now to decide who the winner was, they calculated #(votes)(P(1st)). For example, (127)(18/55) = 41.56 was associated with the possibility that **a** was the winner; (146)(12/55) = 31.8 was the probability that b was the winner; (162)(10/55) = 29.45 was for c; for d was (221)(9/55) = 36 and for d was 20.61. With this idea they identified **a** as the elected teacher. These students argued that the nature of the problem could be interpreted in terms of a "random choice" (selection of a candidate) and based on this relation they decided to introduce the concept of probability. They never explained the nature or rationale of their approach.

Two pairs of students really struggled to understand how the information given in the statement could help them to attack it. They spent significant time trying to see if the data as given in the statement could provide the winner. For example, one pair agreed that there was not enough information to select the winner. The other pair decided to take into account only the information in which the candidates received the first preference. That is, they ignored the rest of the information and chose **a** as the winner. However, it is important to mention that when these two pairs of students were questioned on their reasoning by the interviewer, they were able to solve the problem.

It is important to observe that when the students used the idea of explanation to arrange the votes, they never questioned whether or not the use of different explanations (why not?, for example, -2, -1, 0, 1, 2) could change the order of the information. It seems that they assumed that the order would not be altered if they assigned different ponderings or that they did not need any proof or mathematical argument to use this assumption.

The students who showed significant work on the second problem decided to draw a diagram showing the given information. All the students initially focused their attention on the length and width of the rectangle and used $l = 1.5w$ to represent the relationship between the sides of the rectangle (length and width). They also observed that the perimeter of the inside track could be expressed as $2\,(1.5)\,(w) + \pi w$. It was clear that the use of symbols helped them make progress to the solution.

$w(3 + \pi) = 1500$ was the expression they used to calculate the value of w. That is, $w = 1500/(3+\pi)$ and $l = (1.5)[(1500)/(3+\pi)]$ were given as the dimensions of the first rectangle. What made different the students approaches was the type of assumption they made to calculate the perimeter of the next lanes. For example, two pair of students assumed that the length of the rectangle was the same for each lane, that is, they considered that the straight part of the track was always the same. Thus, to calculate the perimeter of the second lane they stated that $w = 2 + 1500/(3+\pi)$ and wrote that $P = 2l + \pi w$. Here, $P = 3(1500)/(3 + \pi) + \pi[2 + 1500/(3+\pi)]$ and they used a similar procedure to find the perimeter of the other lanes. They also noticed that the general value of $w_n = 1500/(3 +\pi) + 2n$ with $n = 0, 1, 2$, etc. But, these students never realized that they were not using the information that the length is 1.5 times the width for each rectangle; although they used this information in the initial lane. The other four pairs of students applied the relationship between the sides of the rectangle to each lane. However, there were notable differences in their approaches. For example, three pairs of those students found the perimeter of the second, third,...and fifth lane with the next calculations:

$$w_1 = 1500/(3 +\pi) \text{ and } l_1 = (1.5)\,[1500/(3+\pi)]\text{, then}$$

$$P_1 = 2l_1 + \pi w_1 = 3[1500/(3+\pi)] + \pi[1500/(3 +\pi)]\text{; for the second lane}$$

$$w_2 = 1500/(3 +\pi) + 2 \text{ and } l_2 = (1.5)\,[1500/(3 +\pi) +2]$$

$$\text{and } P_2 = 2l_2 + \pi w_2 \text{ that is } P = (3)\,[1500/(3 +\pi) +2] + \pi[\,1500/(3 +\pi) + 2] \text{ and so on..}$$

One pair of students used the following approach to calculate the perimeter of the lanes:

$$w_1 = 1500/(3 +\pi) \text{ and } l_1 = (1.5)\,[1500/(3+\pi)];\ w_2 = w_1 + 2 \text{ and } l_2 = (1.5)\,(w_1 + 2)$$

$$P_2 = 3(w_1 + 2) + \pi(w_1 + 2) = 3w_1 + \pi w_1 + 2\,(3 + \pi)$$

$$w_3 = w_1 + 4 \text{ and } l_3 = (1.5)\,(w_1 + 4);\ P_3 = 3(w_1 + 4) + \pi(w_1 + 4) = 3w_1 + \pi w_1 + 4(3 + \pi)$$

$$w_4 = w_1 + 6 \text{ and } l_4 = (1.5)\,(w_1 + 6);\ P_4 = 3(w_1 + 6) + \pi(w_1 + 6) = 3w_1 + \pi w_1 + 6(3 + \pi)$$

$$w_5 = w_1 + 8 \text{ and } l_4 = (1.5)\,(w_1 + 8);\ P_5 = 3(w_1 + 8) + \pi(w_1 + 8) = 3w_1 + \pi w_1 + 8(3 + \pi)$$

$$w_n = w_1 + 2(n-1) \text{ and } l_n = (1.5)\,(w_1 + 2(n-1))$$

$$P_n = 3(w_1 + 2(n-1)) + \pi(w_1 + 2(n-1)) = 3w_1 + \pi w_1 + 2(n-1)[3 +\pi]$$

This pair of students was the only pair to analyze the generalized case systematically. It

seems that focusing on the symbolic representation rather than numbers calculations made it easier to see the pattern involved in the perimeter of the lanes.

Discussion of Results

To frame the discussion of the students work, it will be important to organize their work around basic questions: Is it possible to trace basic mathematical strategies from the students work? To what extent did the previous learning activities used in the calculus course and the sessions given by the researchers influence their approaches to the problems? There is indication that the two problems given to the students were new for them. It was clear that all the pairs used different means to understand and select a plan to the solution. For example, the students who provided an answer to the first problem, they reorganized the information by using a list or table. For the second problem, the use of a diagram to represent the track helped them to focus their attention on what information was important in the problem. However, it is important to mention that students who directed their work to obtain a final calculation experienced difficulties to analyzing or identifying general properties or relationships among the partial responses. Thus, students who tried to find a number for the perimeter of each lane failed to identify the pattern involved in their answers. In general, they never tried to evaluate their procedures or change their approaches.

In this study, there is indication that the problems used during the instruction helped students to overcome initial difficulties and were eventually able to access and use important strategies. It seems that when the context of the problem, the terms involved, and the general statement are familiar to the students, then they show interest to work on it. The presence of this type of problem during instruction could be an initial stage to motivate students to apply the resources and strategies studied previously.

Finally, when the students were asked to formulate problems or change their original statement, they became engaged in discussions that helped them evaluate key components to get the solution. The work shown by the students indicates that, in general, they focused their attention on the elements that were important in the problems, but they need to develop more strategies that help them reflect on the plausibility of their approaches. If instructors consistently value the students participation in the process of formulating problems and analyze with them their work, then students may be encouraged to develop more powerful tools that help them transfer their knowledge to novel situations.

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TESTING A LEARNING ENVIRONMENT BASED ON A SEMANTIC ANALYSIS OF MATH PROBLEMS: A TWO-EXPERIMENT STUDY[*]

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The two experiments reported here compare three different instructional environments on arithmetic word problem-solving performance of low-ability adults. The respective environments included tutoring, the use of a computer interface that provided students with access to special representational tools, and practice at problem solving. The first experiment compares an environment in which students use hierarchical solution trees to solve problems (TAPS), to one in which students diagram semantic aspects of problems (TiPS). Results showed that students in the TiPS condition did significantly better than those using TAPS. The second experiment compared TiPS to an environment in which students utilized a software tool with an imbedded heuristic designed to guide problem solving (QuEST). Again, students in the TiPS group showed significantly greater gains than those in the comparison group.

Many adult remedial math students find word problems difficult. One explanation is that problem-solvers fail to understand the problems that they confront. Another explanation is that problem solvers often fail to attack problems in a systematic and thorough way--that is, they fail to employ the meta-procedures that expert problem solvers might use. Both explanations suggest that the use of cognitive tools, such as representational forms that aid in problem understanding, or tools that facilitate metacognition, might improve problem solving. The two experiments reported in this study investigate differences in performance when students work math problems and receive tutoring in one of three different computer-based learning environments: one which requires students to build and follow a meta-procedural representation (QuEST); one in which students diagram problem elements and operations using solution trees (TAPS); and one in which students diagram semantic aspects of problems (TiPS).

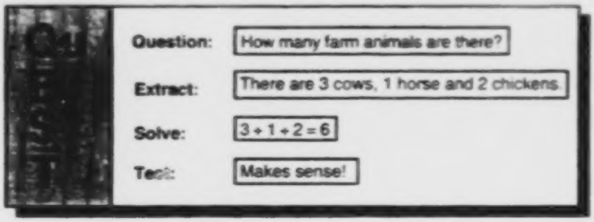
Design Framework

Meta-procedural Representations

Schoenfeld (1983) has identified five qualitatively distinct phases of experts' problem-solving behavior: reading, exploring, analysis, planning and plan implementation, and verification. Thus Schoenfeld determined that experts solve problems according to a heuristic, composed of the five patterns of behavior noted above. Schoenfeld's heuristic is reminiscent of Polya's (1957) conception of mathematical problem solving as a four-step activity composed of understanding, planning, executing the plan, and evaluating results. Proponents hypothesize that heuristics or meta-procedures mediate between the problem space and the cognitive powers of the solver. While heuristics are an emergent phenomena in expert problem-solving, they can be reified as meta-procedures for other (typically non-expert) problem-solvers to follow. We have developed software called QuEST based on these notions. When using QuEST, users identify the question asked in a word problem; extract necessary information from the problem text; create a mathematical expression (which is solved by the system); and check to see if the solution makes sense. Embedding a heuristic in math software is not unique to QuEST. Popular software titles such as MathBlaster Plus by Davidson and Associates, Inc. incorporate similar meta-procedures. A facsimile of the QuEST window appears below:

The Farmer Green Problem

Farmer Green has only chickens, cows, and horses on his farm. If there are 2 chickens, 1 horse, and 3 cows, then how many farm animals does Farmer Green have?



Question: How many farm animals are there?

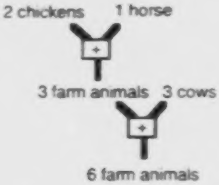
Extract: There are 3 cows, 1 horse and 2 chickens.

Solve: 3 + 1 + 2 = 6

Test: Makes sense!

Schematics based on tree diagrams

The second software tool used in this study is based on a simple schematic representation: tree hierarchies. Similar to Reusser's (1993) HERON system, TAPS lets students construct and label tree diagrams that represent arithmetic relationships existing in the word problem text. The Farmer Green problem is presented in solution tree form:



Reusser (1993) suggests solution trees of this sort make students' thinking overt and accessible and encourage generative understanding (because students can begin work on a problem without having completely understood it). It its hypothesized that they mediate between the various stages of problem solving by retaining important situational aspects in a problem-model form.

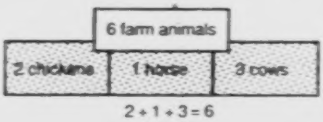
Semantic Representations

Semantic analysis distinguishes between arithmetic word problems according to meaning structures within the problems themselves. Although the two problems below are solvable with the same equation (7-2=5), they represent different semantic types:

1. Marco had 7 grapes, then gave 2 to Meg. How many does Marco now have?
2. Marco has 7 grapes, and Meg had 2. How many more does Marco have than Meg?

The first problem above represents the type called *Separate* (Carpenter & Moser, 1982) or *Change* (Marshall, 1995). This problem situation involves decrementing a particular class of things over time. The second problem represents the *Comparison* type (Marshall, 1995), in which two sets of things are compared. These two types, plus a part-whole or *Group* relation and two multiplicative relations (*Restate* and *Vary*), comprise a set of five relations that account for approximately 95% of the pre-algebra word problems found in math textbooks (Marshall, 1995).

We have developed a software tool called TiPS that allows students to diagram the semantic aspects of word problems. TiPS provides students with a set of graphical tools analogous to the five math relations. Students construct diagrams of word problems that preserve these semantic patterns, while providing a link to familiar arithmetic notation. The Farmer Green problem is presented below in semantic notation:



Experiment 1

The target population for all three learning environments is adult remedial students—those with school-based exposure to arithmetic, but who are generally poor performers in this domain. In our first experiment, we sought to determine whether such students would show greater gains after receiving instruction and practice in a semantic-based environment (TiPS) versus either an environment based on solution trees (TAPS) and one in which students solved word problems without a computer (Practice).

Volunteering university undergraduates were given a pencil-and-paper pretest composed of 4 problems selected to represent a broad sample of the semantic categories. Since we were primarily concerned with treatment effects for adult remedial students, we disqualified high-scoring students from further participation. Fifty-six qualifying participants were then randomly paired and assigned to one of three conditions: (1) TiPS, the semantic environment; (2) TAPS; and (3) the pencil and paper Practice group. Student pairs met for four sessions. Except for requiring correct use of the semantic structures for the TiPS group, the TiPS and TAPS groups were treated the same. The tutor presented a worked example, and then asked the pair to solve a similar problem. After the worked example pairs, the students solved problems until achieving mastery, defined as being able to solve one-step problems regardless of schema type or position of unknown. Both TiPS and TAPS groups worked on the same problems, and the experimenter used the same mastery definitions. The process was repeated with two-step problems after mastery had been achieved on one-step problems. Training time was the same for all dyads.

Results

Following the tutoring and practice sessions, each student was individually administered a pencil-and-paper posttest comparable to the pretest. Since students did not use the computer interface on the posttest, it is reasonable to consider this as a near-transfer test. Tests were graded according to a 4-level rubric that measures the similarity of the student solution to an expert solution. A comparison of posttest means was computed, with the pretest as a covariate, using ANCOVA. Students participating in the TiPS environment performed significantly better than those using TAPS, F(1,25)=4.247, p=.05. Comparing TiPS to the Practice group proved problematic since the variances in these two groups were unequal, F(13,27)=2.18, p<.05, a violation of a basic assumption of tests based on the F distribution. The significantly greater variance in performance for the Practice group is itself interesting.

Experiment 1 Means & Standard Deviations

		T1	T2
TiPS	Mean	9.04	10.43
	Stdev	*.63*	*.76*
TAPS	Mean	8.93	9.79
	Stdev	*.70*	*.89*
Practice	Mean	9.07	9.93
	Stdev	*.76*	*1.58*

Seven of 28 TiPS students used semantic diagrams similar to those in the TiPS interface.

Experiment 2

Next we compared TiPS and QuEST, the meta-procedural environment discussed above. We conducted this experiment in a realistic adult educational environment—a community college's adult learning center. Participants were recruited from its non-credit adult basic education programs.

Because of the difficulties involved in scheduling pairs of participants, individuals were the unit of analysis in this experiment. Volunteers were administered a test which involved solving single- and multi-step arithmetic problems with pencil and paper. Upon evaluating these tests, high-scorers were excused. Students who scored below a minimum threshold were referred to individualized, traditional instruction in the college's adult learning center and participated no further. The remaining 19 students were randomly assigned to either the TiPS or QuEST condition. Students met with tutors for a total of seven sessions. The first two of these were devoted primarily to interface instruction and conceptual knowledge related to problem semantics in the case of TiPS and heuristics in the case of QuEST. The last five sessions involved problem-solving practice, with on-demand tutorial help. As in the first experiment, except for aspects of instruction related to the interfaces, the TiPS and

QuEST groups were treated the same: students worked problems from the same problem bank, regardless of treatment; the same three tutors provided instruction, and the number of problems solved and time spent in training were controlled. Tests isomorphic to the pretest followed the third, fifth and seventh sessions.

Results

We provide means and standard deviations for the paper-and-pencil posttests in the table below. For these data, the appropriate comparison variable is the average of the three tests (T2, T3, & T4) given after treatments were initiated (J. Levin, personal communication, June 21, 1996). A non-parametric test measuring group by sessions interaction showed that TiPS was significantly more effective than QuEST, Mann-Whitney U= 17, p=.022 (Z= -2.294), with an effect size *d*=0.75.

Experiment 2 Means & Standard Deviations

		T1	Avg T2,T3,T4	Difference	
TiPS	Mean	7.50	12.37	*4.87	* $p<0.05$
	Stdev	*2.88*	*1.49*		
QuEST	Mean	8.00	10.48	2.48	
	Stdev	*1.94*	*3.12*		

Two of the 10 students in the TiPS condition used diagrams recognizably similar to the TiPS diagrams, while one of the QuEST students used a diagram similar to QuEST's.

Conclusion

These two experiments provide evidence that remedial adult learners working in a semantic-based environment show greater gains in problem-solving performance than those working in heuristic-based or solution tree-based environments. This is interesting on two fronts. We believe this finding holds practical significance for adult remedial math education, since our experience suggests that heuristics are heavily taught in remedial math settings. Insofar as this comes at the expense of instruction in identifying and diagramming semantic aspects of problems, the instruction may be misguided. The same holds for instruction based upon constructing solution trees.

The second point of interest concerns causes. Why did a semantic environment provide greater benefits than the alternatives? Since relatively small percentages of students carried the TiPS notation over to pencil-and-paper problem solving, it is difficult to ascribe the difference to the use of these. However, using the TiPS tools during instruction and practice may have supported certain patterns of thought on the part of students, such as discriminating semantic aspects of word problems, allowing students to construct more meaningful representations during posttest problem solving. A second explanation, not inconsistent with the first, is that the use of the TiPS tools had an impact on the human tutorial interaction that occured across sessions. The notion that the social and material environment in which learning occurs has an effect on learning is consistent with the socio-cognitive perspective of learning theorists (e.g., Vygotsky, 1978; Rogoff, 1990). This calls for further investigation of how the quality of tutoring varies with environment.

File
Edit
View
Window
Help
Del
Done
All Clear
Total
For Help, press F1
NUM

screen shot of TiPS Screen

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PROGRESSION IN WORD PROBLEM SOLVING BY DRAWING AND EXPLANATION

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Thirty first-graders in a midwestern, mostly Latino, low SES, urban school participated in a 2-month fall teaching experiment, focused on children's solving of word problems. Two observers recorded students' solution methods and explanations. The teacher created a mathematics classroom rich in invented strategies and explanatory discourse by sending five students to solve problems at the board while the rest solved at their seats. Students were actively engaged in both solving and inventing their own problems.

Students were encouraged to draw the quantities in the problem situation and to operate on these quantities to solve the problem. The drawings were then available for other students to reflect on, and children pointed to parts of the problem as they explained their solution. Individual problem-solving progressed from 25% to 90% of the class correct on simple word problem types and from 4% to 50% correct on middle-difficulty word problems (more difficult than are often given in first grade). Over time, children saw a range of solutions for each kind of problem. Through teacher scaffolding, modeling by more expert students, and help from peers at the board and at seats, explanations of most students moved within the following progression: (a) from simply stating or guessing answers to (b) pointing out on their drawing parts of the problem situation with teacher help to (c) giving the problem situation and pointing out parts of their drawing on their own or (d) describing actions taken to solve the problems (c and d were each done first by some children) to (e) an integrated verbal and indicatory explanation of how the situation was represented and then acted on in order to solve the problem. Most students at least developed to stage (c) or stage (d).

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RATIONAL NUMBERS

PRESERVICE ELEMENTARY TEACHERS' UNDERSTANDING OF DIVISION WITH WHOLE NUMBERS AND FRACTIONS

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Fifty preservice elementary teachers provided written, inventory, and coursework data about their prior understanding and developments in understanding about division with whole number and fractions during an elementary mathematics methods course. Although 82% of the preservice teachers could *recognize* partitive and quotitive division story problems involving fractions, understanding about division involving either whole numbers or fractions that would enable these preservice teachers to discuss division as a fundamental structural base within the multiplicative conceptual field seemed disarmingly weak. Six structural dimensions of understanding—or needed structural dimensions of understanding—about division occurred and recurred among the preservice teachers as they revealed or constructed understanding.

Introduction

In order for elementary mathematics teachers to be able to support elementary students in constructing robust understandings about the four fundamental arithmetic operations and their direct functions within the conceptual frameworks of "additive structures" and "multiplicative structures," these teachers need, themselves, to have robust understandings of these operations. Research conducted from the 1970s forward reveals that elementary and preservice teachers often exhibit understandings that are not robust regarding the operations of multiplication and division, which form the fundamental structure of the multiplicative structures or "multiplicative conceptual field." Multiplication and division form the structural foundation for proportional reasoning, reasoning that is the "capstone" of elementary mathematics and "cornerstone" of advanced mathematics (Lesh, Post, & Behr, 1988, p. 94).

Purpose and Theoretical Framework

The current study investigates preservice elementary teachers' understanding of division involving whole numbers and fractions. The theory guiding this research is the theory that construction of understanding of the conceptual field of multiplicative structures is both an extended process and one that requires reconceptualization in the rational number domain (Greer, 1994; Harel & Confrey, 1994). Previous research has demonstrated limitations in preservice elementary teachers' understandings about division with whole and rational numbers. The current research further investigates the understandings and structures of understandings about division that preservice elementary teachers *have* developed and continue to develop as they construct understanding of multiplicative structures in the whole and rational number domains.

Research Questions

The research questions guiding the research were: (1) What understandings about division with whole numbers and fractions do the preservice elementary teachers in this study (who have completed a prerequisite course in elementary mathematics content regarding number systems) demonstrate as they enter the required elementary mathematics methods course? and (2) What structural dimensions of understanding — or needed structural dimensions of understanding — about division with whole numbers and fractions occur and recur among the preservice teachers as they develop and/or continue to develop understanding about division?

Methodology

Fifty preservice elementary teachers enrolled in a university elementary mathematics methods course volunteered to complete a written inventory requesting that they (a) construct and solve story problems for given division expressions with whole numbers and fractions, (b) identify and explain the remainder and fraction forms of quotients in the story problems created [in part (a)] for whole number division expressions, and (c) given story problems involving fractions modeling addition, subtraction, multiplication, and division, identify story problems modeling division and solve these story problems. In addition to the written inventory, each preservice teacher was interviewed about his/her answers on the inventory to further elicit structures of understanding about division. Following the inventory and interviews, instruction about conceptual models for division with whole numbers and fractions was provided by the researcher/teacher. One further interview with each preservice teacher was conducted following this instruction. During this interview, the preservice teachers were asked to describe their current understanding of division involving whole numbers and fractions, and to discuss the developments in their understanding during the course. (Division expressions involving whole numbers and fractions were given as prompts during this interview.) During the course, coursework related to division was also collected and used as data for the study. Inventory, coursework, and interview data were interpreted to determine structures in the responses and describe entry-level understandings as well as structural developments and needed structural dimensions in understandings of the preservice teachers.

Findings

Although 82% of the preservice teachers in the study could *recognize* story problems modeled by division (involving fractions) — by correctly identifying the two story problems directly modeled by division expressions among five story problems modeled by all four operations, understanding about division involving either whole numbers or fractions that would enable these preservice teachers to discuss division as a fundamental structural base within the multiplicative conceptual field seemed disarmingly weak. Three of the 50 preservice elementary teachers entered the course communicating that they had personally constructed two meanings for division, meanings which were consistent (for each of these three students) with Partition and Quotition division interpretations: Partition involving dividing a quantity into a given number of equal partitions with the result of the division revealing the size of each partition and Quotition involving the division of a quantity into measures of a given size, with the result of the division revealing the quantity of measures of the given size achieved. The other 47 preservice teachers, through the Inventory given at the start of the course and the interview about their Inventory answers, revealed that they had not prior to this experience categorized their thinking about division into more than one category. Entry level data (data based on the Inventory and interview about the Inventory) are provided in Tables 1, 2, and 3 below.

Table 1. Percentage of teachers who, given the division expressions below involving whole numbers and fractions, were able to construct and solve a story problem modeled by each division expression (n = 50).

Division expression	57 ÷ 8	34 ÷ 4	$\frac{1}{2} \div 4$	$2 \div \frac{1}{4}$	$\frac{3}{4} \div \frac{1}{2}$
% of preservice teachers constructing & solving a story	100%	96%	72%	56%	4%

Table 2. Percentage of preservice teachers who, given five story problems involving fractions that were modeled by +, −, x, and/or ÷ expressions, were able to recognize and solve the division story problems [one Partition and one Quotition] (n = 50).

Division Story Problem Type and Expression	% recognizing each story modeled by division	% solving each division story problem
Partition story: 1 1/2 ÷ 3	92%	82%
Quotition story: 2 ÷ 4/5	82%	68%

Table 3. Percentage of preservice teachers who were able to provide and interpret a fraction form — rather than only a remainder form — for the result of the division of 57 ÷ 8 and 34 ÷ 4 using the two whole number division story problems they constructed for these expressions (n = 48).

	Provided & Interpreted 2 Fraction Quotient Forms	Provided & Interpreted 1 Fraction Quotient Form	Provided & Interpreted 0 Fraction Quotient Forms
% able to provide & interpret the fraction form	46%	24%	30%

Analysis of the data from the Inventory, both interviews, and the coursework revealed a great number of areas of unclear understanding or complete unawareness of dimensions within the structure of division among the preservice teachers. As the preservice teachers constructed or continued to construct understanding about division, structural dimensions of understanding — or needed structural dimensions of understanding— occurred and recurred in their developmental processes. These structural areas are briefly described below. Preservice teachers with better developed understandings revealed greater evidence of the structural dimensions described below. Preservice teachers with less developed understandings revealed incomplete or undeveloped understandings of these structural components.

The structural dimensions or needed structural dimensions of understanding about division with whole numbers and fractions were as follows (not hierarchically-ordered):

1. Fully conceptualizing partitive and quotitive division (described earlier) as distinct division interpretations. Preservice teachers who were able to fully distinguish between partitive division and quotitive division with whole numbers used this distinction to make sense of division with fractions (which they sometimes converted to decimal numbers to support their sense-making). Preservice teachers who had not distinctly conceptualized partitive and quotitive division experienced greater difficulty in re-conceptualizing division with fractions. One-fourth of the preservice teachers in the study communicated in interviews that learning to distinguish the two division interpretations was extremely difficult for them (and difficult to varying degrees for a another one-fourth of the preservice teachers).

2. Understanding the distinct roles of dividend and divisor in division situations, how these roles relate to the written symbolism of division ($a \div b$, $b\overline{)a}$, and $\frac{a}{b}$), and how these roles and the written symbolism relate to informal and formal mathematical language of division. Understanding about the distinct roles of divided and divisor in division situations and/or the relationship between these roles and the language of division, both word and notational, regardless of ability to distinctly conceptualize partitive and quotitive

division, was limited in some form in two-thirds of the preservice teachers as they entered the course. Approximately one-fourth of the preservice teachers some or all of the time reversed word usage or mathematical notation such as: $24 \div 4$ and the expression "twenty-four divided *by* four" with whole numbers (instead writing $4 \div 24$ for "24 divided by 4" or reading "four divided by twenty-four" for $24 \div 4$), and these errors persisted with fractions. Similarly, translating from the symbolism $24 \div 4$ to $4\overline{)24}$ or $4 \div 24$ to $24\overline{)4}$ caused difficulty for approximately one-fourth of the students at the start of the study.

3. Distinguishing between incomplete division (division with remainders) and complete division (division without remainders or with non-whole number [fraction or decimal] quotient forms), and understanding how to interpret the results of incomplete and complete division including the relationships between remainder and non-remainder forms of complete division results. The majority of the preservice teachers, upon entering the course, could not both provide and explain a fraction quotient form for the answers to their division story problems for $57 \div 8$ and $34 \div 4$ (expressions for which all but two preservice teachers could write story problems). Learning to relate the meanings of the remainder and fraction forms of quotients in non-integer quotient complete division situations was less difficult using the partitive division interpretation than using the quotitive division interpretation for three-fourths of the preservice teachers. Reconstructing understanding about incomplete and complete division in the domain of fractions (where incomplete division answers were achieved through conceptual rather than procedural means, such as using fraction area models to determine that $4 \div \frac{3}{4}$ results in 5 quantities of $\frac{3}{4}$ totaling $4\frac{3}{4}$, with $\frac{1}{4}$ leftover, which accounts for the complete fraction quotient $5\frac{1}{3}$) was also *difficult* for three-fourths of the preservice teachers.

4. Understanding the connection between multiplication and division in the domain of whole numbers and (re-conceptualized) in the domain of rational numbers. Preservice teachers who discussed division in terms of its connection to multiplication with whole numbers (such as in thinking about $34 \div 4$ as signifying the number that would be multiplied by 4 to equal 34) drew on this connection to address new dimensions of learning about whole number and fraction division.

5. Having learned — being able to remember and apply correctly — an algorithm for dividing fractions. The preservice teachers who, during the course, had access to an algorithm for dividing fractions used their procedural knowledge to provide answers that could support their attempts to develop meaning for fraction division. Approximately one-sixth of the students explained that they had never been able to remember the division

algorithm they had learned because of having difficulties remembering which numbers to multiply with which numbers (e.g., remembering which numbers to multiply and where to position the products when taught to multiply a x b and place this product "over" the product b x c when vertically setting up fraction division: $\frac{\frac{a}{b}}{\frac{c}{d}}$) Two preservice teachers learned to "invert and multiply" but inverted the dividend; one preservice teacher inverted either dividend or divisor, believing that it did not matter which fraction was inverted. Other preservice teachers who were confident in their knowledge of a fraction division algorithm, whether or not they had previously thought about with the concepts modeled by fraction division, were able to use their procedurally-acquired answers (in which they had confidence) to tackle the problem of making sense of the concepts modeled by the division expressions.

6. Understanding the role of referents (units of measure) in situations modeled by division. Preservice teachers who discussed division contexts with whole numbers in terms of the "referents" (units of measure) referred to in dividend, divisor, and quotient, drew on this understanding to help clarify meaning for fraction division. This understanding usually was observed in conjunction with understanding of the two distinct division interpretations, partition and quotition. For preservice teachers not secure in their understanding of the role of referents in whole number division, the issue of referents could compound the difficulties in making sense of fraction division.

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DEVELOPING CHILDREN'S RATIONAL NUMBER SENSE: A NEW MODEL AND AN EXPERIMENTAL PROGRAM

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A new model for the psychological structure for the development of rational number understanding was proposed and an experimental instructional program was devised and tested. It was proposed that the coordination of children's intuitive abilities at proportional assessments and their 'splitting' schemas form a core conceptual organizing structure in this domain. Sixteen grade four students participated in the experimental curriculum. A comparison class of 18 students of similar demographic characteristics and matching mathematical abilities received a traditional curriculum. Students from the experimental group were more successful in all aspects of rational number performance. Flexibility of movement among representations, understanding of magnitude, use of percent benchmarks and the halfing strategy were general characteristics that emerged.

The domain of rational number has traditionally been a very difficult one for students to master. A substantial amount of research has been devoted to understanding and improving this situation. Kieren has suggested that this research has had two main thrusts: epistemological and psychological. The epistemological thrust has been devoted to clarifying the nature of rational numbers as mathematical constructs, and the "sub-constructs" of which they are comprised (Behr, Harel, Post, & Lesh, 1992). The psychological thrust has been devoted to identifying the schemas that children bring to the field of rational number and the way in which these schemas develop as they gradually master (or fail to master) the field (Case, 1978, 1985; Confrey 1994; Keiren,1994; Steffe 1994).

A number of teaching studies have emanated from this research. Although these studies have generated gains in controlled circumstances, problems still remain. As a result, it has been suggested that more attention needs to be placed on students' integrated use of the entire rational number domain in order to harness the gains in subconstruct knowledge that instructional research has produced (Post , Cramer, Behr, Lesh, and Harel, 1993). Particular attention should be paid to the different ways in which students represent the rational numbers in different contexts and to their ability to move among these representations in a flexible manner (Sowder, 1994).

The present study had its origins in the psychological tradition mentioned above. Its purpose was to analyze children's global understanding of the rational number system in such a fashion as to develop a curriculum which would produce the sort of general understanding and flexibility of representation called for by Post, et al. and Sowder. The theoretical framework that was used for analyzing children's general conceptual understanding was the neo-Piagetian one developed by Case and his colleagues (Case et al. 1992). In previous work, this framework was applied to modeling children's understanding of the whole number system (Case & Sowder, 1989; Case & Okamoto, 1996). This analysis, in turn, was used to create a curriculum to develop improved whole number sense, including greater flexibility of representation (Griffin, Case & Seigler 1994). In the present study, the same approach was applied to modeling the development of children's understanding of the rational number system.

Psychological Analysis

The basic hypothesis was similar to one proposed by Case (1985) and Resnick and Singer (1994): namely, that children's understanding of whole and rational number develops in a formally similar way. In each case, children's numerical and global quantitative schemas develop separately at the outset. As they make the transition to a higher level of cognitive development, children gradually coordinate these two schemas to yield a core understanding of the way in which the "easy" (e.g. low) numbers in the field in question are structured. This core understanding is then extended to harder (e.g. "higher") numbers, until the structure of the entire field is understood. What is different in the development of whole number and rational understanding is the nature of the original units, and the stage of development at which they are first coordinated: in the first transition, which occurs as children enter the stage of concrete operations, the two "primitive" schemas are those for counting and global quantity comparison. Coordination of these two structures yields a "mental number line" at about age 6. Coordination of different mental number lines with each other yields an understanding of the part-whole structure of addition which is solidly in place by the age of about 8. Finally, this understanding leads to a conceptualization of the part-whole grouping that underlies the base-ten system, and is gradually extended to include the entire field of whole numbers by the age of about 10 years (Resnick, 1983; Case & Okamoto, 1996).

In the later transition the primitive structures are that for "splitting" or "doubling" (Case, 1985; Confrey, 1994; Kieren, 1994) and the structure for proportional evaluation, both of which appear to be in place by about 9-10 years (Case & Sandieson, 1988; Resnick and Singer 1994). Coordination of these two structures at the age of 11-12 yields the first semi-abstract understanding of relative proportion, which is then coordinated, very gradually, over the next several years to yield an understanding of the way in which proportional and additive thought can be merged (as in the decimal system). A full understanding of the way in which ratios, decimals, fractions and percents are related is ultimately acquired.

Although it has been suggested that there is a necessary order in which these different subconstructs must be understood, our own intuition was closer to that of Confrey who believes that the order is more arbitrary, and that what matters is that the general sequence of coordinations remains progressive. As a result, what we attempted to do was to produce the greatest breadth and depth of understanding for the least effort. We felt that this could be achieved if children's first coordination was between the halving structure and a proportional analysis of relative "fullness", and their second coordination was between ratios, so understood, and percents. Our reason for this latter assumption was that the additive structure of percents is the same as the field of whole numbers from 1 to 100, which children master during the first two or three grades of school. By expanding children's understanding of percents, we felt that we could promote a reasonable first understanding of the decimal number system, and then use that as a basis for refining their understanding of fractions, and for introducing exercises in which children had to move in a flexible manner between thinking in terms of percents, fractions, and decimals. This present study had three purposes: (1) To re-conceptualize the psychological structure that children develop for understanding rational number; (2) To develop an experimental curriculum that promotes the development of this structure; and (3) To test the impact of this curriculum by assessing changes in the understanding of children who were exposed to the curriculum, as compared to those who encountered a more traditional approach.

Methods

Thirty-four grade four students participated in this study. The experimental group was composed of sixteen grade four students from a laboratory school located at the University of Toronto. The comparison class of eighteen grade four students, which served a similar population, was drawn from a private school near the University. The two schools were founded along similar principles, and shared many features, including strong academic programs, small classes, and individualized attention and programming.

Design

A rational number test comprising 41 items on percents, decimals, and fractions—coupled with detailed protocol for questioning children on their reasoning—was developed. Some items on the test were chosen because they were standard types of school tasks (e.g. what is .5 + .38?), others because they permitted us to assess the breadth and flexibility of children's representations (e.g., what is 1/8 as a decimal?). Many were selected from the literature on mathematics education. This test was administered in an interview format to all students on an individual basis as a pretest and then after the intervention as a post test. The students responses were recorded verbatim. Twenty, forty-minute experimental sessions were taught approximately once a week over a 5 month period by one of the investigators. The control group received a total of twenty-five, forty minute lessons on rational number, but concentrated during a 12 week period toward the end of the year.

Experimental Curriculum

The experimental curriculum began with exercises that were designed to ground students' introduction to rational numbers in a spatially intuitive and perceptually accessible way of representing ratios. They were also designed to promote the coordination of students' natural multiplicative intuitions regarding ratio with their intuitions regarding halving and doubling. The initial context was one in which children had to estimate the degree of "fullness" of different props. The props were cylindrical beakers containing varying amounts of water, and large drainage pipes of assorted lengths that were partially covered by a moveable venting tube. Both these props provided a "side view" that made it easy to draw the proportion of the total object that was covered, using a "number ribbon" style diagram (i.e., a narrow vertical rectangle with some part of the bottom shaded in). In estimating and computing percents, children's natural tendency was to use a halving strategy, and this was encouraged (e.g., Where would the 50% line be on this beaker? And where would 50% of the bottom half be? Now the water comes up to about half of that. What would half of 25% be?). By introducing rational numbers through the teaching of percents we were able to capitalize on students' everyday language, and the knowledge about percents that they had derived from popular culture (Parker and Leinhardt, 1995). We also allowed them to take advantage of the strategies that they spontaneously invented for working with percent computation (Lemke & Reys,1994), many of which involve a spontaneous and appropriate movement between additive and proportional strategies. For example, children seemed to easily grasp the notion that 75% could be decomposed into 50% and 25%.

Once children were comfortable in estimating, computing, and talking about percents, we introduced them to two-place decimals in the context of a "walk." Large laminated number lines were placed on the classroom floor. Students walked on a number line stopping at some percent of the total distance. By explaining that decimal numbers indicated the "percent" of the distance that one had traveled, we were able to give children a first intuitive understanding of this new form of representation. Thus, if a child had passed the 2 meter mark on the walk, and gone some way toward the 3 meter mark, she would be encouraged to figure out that value as a percent (e.g. 25%) and then express the total distance as a decimal (2.25 Meters). Once again, children seemed to find this translation a natural one, in this context. Children also played games using LCD stopwatches with screens that displayed seconds and hundredths of seconds, and these were interpreted as temporal analogs of distance. The numberline exercises were continued through the playing of board games, where moves in the games consisted of adding and subtracting these decimal amounts. Fractions were taught throughout the program in relation to decimals and percents. At the beginning, for example, children naturally used the term one-half interchangeably with 50%; we also introduced the term one-fourth. Finally, toward the end of the program, exercises were presented that required interpreting quantities through all three forms of representation. The general style of teaching throughout the experimental program was one that encouraged children to look for patterns, and to share their discoveries with each other. Students worked in large groups taught by the researcher, small student-run groups, or pairs. At certain points in the curriculum, they discussed what they had learned and prepared new materials to help teach younger children. At no point were students taught any formal algorithms.

Control Curriculum.

The control group devoted an equal time to the study of math, and followed the program for rational numbers from the widely used Canadian math text series, *Math_Quest 4* (Kelly, et al., 1987). The sequence of this grade four program started with fractions, where fractions are initially defined as numbers that describe parts of a whole and are represented by pie chart diagrams. Fractions of a set, equivalent fractions, and comparing fractions followed. Decimals were taught next using pie graphs, number lines and place value charts. Tenths were introduced and their relation to decimals was shown. Finally, equivalent decimals were taught by showing that numbers such as .3 and .30 were equivalent because 3/10 is the same as 30/100. Ten lessons on operations with decimals followed. A rule based format was used to teach addition and subtraction of decimals as well as multiplication of one and two place decimals. The classes in rational number ended with computation with fractions and division of decimals. Large and small group activities were standard practice in this classroom. A wide variety of manipulatives were used and discussion was promoted. The teacher was enthusiastic about mathematics, and considered it one of her priorities. Thus the primary difference between the two groups was in the nature of the curriculum.

Data Analysis and Results

In order to analyze the tests, items were grouped into six subcategories. Five of these categories have been shown in the literature to indicate rational "number sense" and conceptual understanding. These subcategories were: 1) Use of rational number representations interchangeably and flexibly; 2) Understanding of magnitude of rational numbers and ability to compare and order these numbers in their different symbolic representations; 3) Overcoming visual perceptual distractors; 4) Performance of non-standard algorithms by inventing strategies; 5) Ability to perform real -world type problems. The sixth category assessed students' ability to perform standard algorithms. The study was analyzed both quantitatively and qualitatively. There were no significant differences between groups in pre-test mean scores. Both groups showed an improvement in total test score in the course of the study. However, the scores of the experimental group improved much more than those of the control group. Thus, when a two way analysis of variance with repeated measures was conducted, a strong interaction effect emerged for group x time of testing $F(1,32) = 32.49$ $p < .001$. The same pattern of results was evident for five of the six subcategories mentioned above. This was not found for the subcategory standard algorithms. On this subtest there was no significant difference in the performance of two groups.

Qualitative analyses showed that the form of explanations given were never rote or algorithmic. Flexibility in moving among representations and an understanding of general magnitude relationships were general characteristics that emerged repeatedly. For example in response to the item, "What is 1/8 as a decimal?", a student from the experimental group responded 0.125 Experimenter: "How did you get that answer?" Student : "Well, 1/4 is 25% ... and 1/8 is half of that so it is 12 1/2/% ... so 12 1/2% is .12 1/2 or .12.5 ... no I think it is just .125". Further, the students showed a consistent use of percent benchmarks to interpret both decimals and fractions, and persistent use of the halving strategy. The students' responses to the test item "What is 65% of 160," exemplify how they used these benchmarks to compute challenging problems. One student asserted:.."50% (of 160) is 80. I figure out 10% which would be 16. Then I divided 16 by 2 which is 8 (5%), then 16 plus 8 um ... 24. Then I do 80 plus 24 which would be 104". Another student responded as follows "50% of 160 is 80 .. 25% is 40 so 75% (of 160) is 120 so it would be a little less than that (120) it would be 10% less so it would be about 108". Even when they made errors, the conceptual understanding of the experimental group often seemed solid. In contrast, even when they succeeded, the conceptual understanding of the control group often seemed shaky, and many of their errors seemed to stem from a rather unreflective application of a standard procedure (e.g. shading in 3 of 8 parts of a pie-segment

Conclusions

The results are encouraging. In our talk, we will provide examples of students' reasoning from our protocols, and relate them to our general theory of conceptual development.

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TEACHING OF FRACTIONS FROM A PHENOMENOLOGICAL POINT OF VIEW

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After the recent curricular changes which reorganized the bases of mexican elementary teaching, we carried out research in a classroom, whose problem was the role that the use of drawings plays in the transition from word language to mathematical language, in the scope of fractions. From the hypothesis that drawings contribute that transit from word language to mathematical language, we developed a teaching program for fourth grade (based on a "didactical phenomenology" focus and on recent didactical proposals for the classroom), and applied a pre-test and a post-test to an experimental group and to a control group. The recorded results allowed us to confirm the hypothesis (leads by three non-parametric tests) and to bring about a qualitative analysis of the teaching experience and of the interchange which propitiated in the group in which it took place.

Introduction

The recent *change in the official programs in 1993*, in *elementary schools* in Mexico, has propitiated new questions that must be answered through research in the classroom. These changes endeavor the development of proper didactical strategies that help students to generate rich meanings linked to the curricular contents undertaken. Thus we have focused some *concrete instructional models associated with the use of drawing, representation of fractions and the appearance of valuable translation processes requested by the joint use of different languages*.

The general frame we have set for such experience has been that of "didactical phenomenology" (in the sense given by Freudenthal, 1983). That is, we have chosen a teaching program consisting of plenty of models and meanings, whose evident starting point is sustained in diverse concrete situations which recreate the phenomena recognizable by students, situations which also propitiate a subsequent progress towards less perceptive mathematical entities.

A relevant *theoretical background* when posing our *didactical process* was the recent contribution of Streefland (1993), who has proposed teaching models linked to *actual situations* and *strong visual aids* in order to encourage the *construction of a language for fractions* and at the same time to *develop exercises that make easy the construction of certain thinking models* (Streefland, 1984a, 1984b). The stress put by this researcher on strengthening and following didactically students' works provided us with a relevant constructivist didactical frame for our teaching program.

In regard to basic semantic contents incorporated in our didactical process, we favored the meanings of *measure*, *quotient* and *ratio*, attending to its intuitive character and the reciprocal potential links that Kieren (1984, 1985, 1988) has admitted. Also we leaned on the very elemental notion *"fracturing operator"* that Freudenthal (1983) has identified in situations related to the recognition of the *"n-th part of"* a magnitude or an object.

We lean on the basis that Sastre (1984) and Valdemoros (1994a, 1994b) confer to *the use of drawing in teaching context*, taking into consideration its *concrete graphical representation* character (this is, according to features of the represented objects) and its efficiency in proportioning a transition toward *established arithmetical notations*, which cannot only be introduced but also *be given sense* as the use of specific *empirical teaching models* (such relationships do not emerge spontaneously but through instruction).

The above is related to statements posed by Janvier (1987), Goldin (1987) and Lesh, Post & Behr (1987) regarding inherent *difficulties in translations* involved in the *transition from one representation system to another* (such difficulties obstruct and misinterpret the construction of meanings). The above mentioned researchers agree that those processes of translation have frequently been ignored in the teaching context. The present study has paid special attention to such processes, favoring the enriched role drawings may accomplish.

Within such frame, we identified as a *problem the role that the use of drawings plays in the transit from word language to mathematical language in the scope of fractions* and thus we have formulated the following hypothesis: *drawings help the transition from word language to mathematical language.* In order to prove this, we designed a study of fourth graders in elementary school which includes a teaching program and the application of pre and post tests to both experimental and control groups.

Method

Participants in this study were fifty 9-11 year old students in fourth grade in a public elementary school in Mexico City.

The *independent variable* was the *drawing applied to problems of fractions* while the *dependent variable* was the *corresponding written arithmetical expression*. Definitions of variables are due because when testing, drawings were shown (associated to a text) by researchers while students confer, then a specific written arithmetical answer.

The quasi-experimental design adopted consisted in the application of *pre* and *post tests* to an experimental group (formed by two school groups) and to a *control group*. Both tests consisted of 19 parallel tasks with similar structures and difficulties (four of them referred to *preservation of the unit*, five *fraction recognition* tasks related to diverse semantic contents, four to identification of relations of *equivalence* and six to relations of *order*). The developed experimental process consisted of 17 sessions in which relations and meanings mentioned before were discussed through diverse concrete models, derived from daily experience. Drawing was introduced as a means for translating considered situations in the graphic context and its relation to established notations in the last resort.

The *procedures* applied for selection consisted of randomly choosing a school to carry out the investigation (based on the directory of a wide zone of the city). The school permitted the access to three groups of fourth graders, in the end taking two of them (with

19 and 15 students in each group) both identified as experimental groups and another one (with 26 students) the denominated control group.

We performed a *quantitative analysis* of the results through the integration of non-parametric tests (Wilcoxon's range, Kruskall-Wallis' significance and U Mann-Whitney's test). The qualitative analysis of such information would be focused in teaching sessions and in the main cognitive tendencies that tests had shown. In particular, to carry out the first of these two qualitative tasks of analysis, endows the present work with a mixed feature: it is quasi-experimental research, and at the same time is compounded by relevant situations generated in the classroom which must be undertaken within a qualitative frame. The last components of this experience are of great relevance (Jiménez, 1996).

Results Of Tests

Results of the pre-test made evident a very deficient performance in subjects of the experimental group, who were surpassed by those of the control group, as well as performance of the first was reverted in the post-test.

Wilcoxon's test showed highly significatant differences (0.0000) between pre-test and post-test for all experimental subjects (from school groups A and B); and for the control group such differences between tests took a probability of 0.6155.

In Kruskall-Wallis' test of significance, middle ranges of the pre-test showed an average of 17.13 in school group A; 26.37 in school group B (both experimental subjects) and 34.63 in school group C (control group). In the post-test, averages of middle ranges were 29.61 in group A; 30.90 in group B, and 15.56 in group C. The significance of the levels obtained were 0.0019 in the pre-test and 0.0041 in the post-test. Differences observed after comparison between experimental and control groups have been caused by our didactical process in the first group.

In the U Mann-Whitney's test, the experimental group obtained an average of 21.21 (lower than the control group) in the pre-test, while in the post-test they obtained an average of 30.18 (higher than that obtained by the control group). Here, differences between groups were significatant (0.0009).

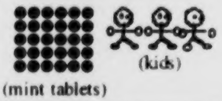
That is, contrast between both tests and comparison between groups, in the light of non-parametric tests before mentioned, showed that experimental subjects exhibited a notably meaningful progress once the development of the teaching program took place (and difference among tests was not significant in the control group). This proves the hypothesis of the present study.

Teaching Experience

Mathematical content approached through teaching were elemental and introductory so that they granted the construction of diverse concepts linked to previously specified meanings of fractions (quotient, measure, ratio and *"fracturing operator"*).

Next we explain some of the situations presented in class:

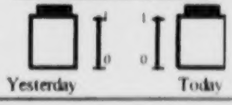
Distribute in equal parts.



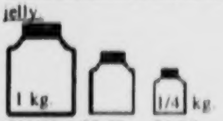
Meaning of quotient. Partitioning situations with pizzas, cakes, mint tablets, chocolates, etc.
Example: Sharing out of mint tablets.

Meaning of measure. Fractional measures in different context of measurement.
Example: Comparison between two different measures.

Yesterday, Jerry drank 3/5 of a milk liter bottle. Today he drank 3/8 of a milk liter bottle. Paint it.



Write the missing weight of jelly.



Meaning of "external ratio". Situations of linking different magnitudes (with elemental proportional variations).
Example: Complete the following scale.

The adopted manners of working included some common features to all sessions. The teacher posed the problematic situation to be interpreted and solved; and then children individually started their tasks of representing the situation using different kinds of concrete materials (modeling clay, diverse objects, strips of paper and cardboard, boxes, bottles, chocolates, etc.). Initial activities were open to confrontation and discussion among children; they were asked to fracture continuous and discrete wholes which were submitted to diverse strategies of subdivision. As activities progressed, children were asked to form a group in order to continue working and be able to face situations of increasing complexity. Once groups stabilized their answers and reached agreements, they were asked to look for different alternative ways for graphical representations (e.g., drawings, outlines, very simple tables).

Unlike applied tests, drawing was chosen by students and its subsequent evolution was subjected to agreements of the group. Only at the end of this progress, the transition to conventional symbolization was introduced.

The teacher posed the problems and inherent questions, always waiting for children to give their own answers. When any contradiction or dispute emerged, she asked them to use the materials available to solve them and then to translate it into graphical representation (by means of different ways of drawing which must by explained and justified). Thus, some simple conventional notations were introduced. Finally, both the group and teacher expressed their agreements, corrections and conclusions.

Students actively participated during the work time. Initial passiveness in some of them was promptly abandoned, in a patent way. In general, members of the experimental group exhibited a greater understanding of the meaning of fractions and a higher performance in the use of graphical representations.

Discussion

From all the above, the hypothesis about the use of drawing favored the transition from word language to mathematics language has been proved.

In the context of a global qualitative analysis of the experience, we want to stress the semantic richness achieved through the working sessions dedicated to teaching. In these, children started their participation regarding outlined problems in classroom, leaning on the use of a great variety of concrete materials, expressing their thoughts in their groups and defending collectivized statements before their classmates. Once this level of progress around "the concrete" was achieved, drawing was introduced as a means of verification and as "memory" of what was constructed in other representation contexts. Established arithmetical notations were later discussed. Frequently, group work was followed by certain kinds of individual work, according to the concrete model.

Given such modalities of interaction and the progress in the classroom, the post-test presented relevant individual cognitive progresses. Particularly, students expressed better and more finished arguments in their answers, and at the same time showed evidence to give a major consideration to corresponding drawings. Initial mistakes, when focusing on the dimension of a figure, patently decreased. Relations of equivalence and order were strongly straightened as were comparisons between fractions, established in drawings.

Conclusions

Within the context of teaching and after manipulating diverse concrete materials, children made an adequate use of drawing as mediating in processes of translation from common language into the arithmetical one, since they solved very elemental problems regarding the identification of fractions as well as to set up equivalence and order relations. Thus, drawing also played an important role in the generation and preservation of the meanings involved.

Tests and their statistical analysis allow us to assert that the use of drawings favored the passing from word language to the mathematical one, after the development of our teaching program.

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USING CONCEPTS AND CONNECTIONS TO TEACH STUDENTS ABOUT DECIMAL NOTATION

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In this paper, we will report on a teaching experiment which took a very different approach to the instruction of decimal notation. Rather than have students build from concrete referents to more abstract concepts, the decimal instruction in this study attempted to help students build a broader abstract set of concepts to get at the more specific, concrete symbol systems involved with decimal notation. Specifically, students were given extensive experience with the broad construct of ratio in order to help them make better conceptual sense of both fractional and decimal symbol systems. Results from this experiment indicate that using this approach helped students development a deeper understanding of decimal notation.

Introduction

The difficulties students have understanding decimal notation have been well documented in the research literature (Carpenter, Corbitt, Kepner, Lindquist, & Reys, 1981; Grossman, 1983; Resnick, Nesher, Leonard, Magone, Omanson, & Peled, 1989; Wearne & Hiebert, 1988, 1989; Hiebert, 1989). Typically, the problems students encounter in their understanding of decimal notation have been viewed as decoding issues. To address these problems and to help students develop deeper conceptual understandings of decimal numbers, the elementary school mathematics curriculum often focuses on helping students "build up" their understanding of decimal notation through the use of concrete referents. Materials such as Dienes base ten blocks and money are often used to help students define decimal numbers more concretely.

However, while using concrete referents will benefit students in some ways, a major problem with some of these "referent-building" activities is that they are often narrow and limited. In many cases, concrete referents are meant to lead to only one specific mathematical construct. For instance, work with the Dienes blocks is often intended to provide specific and direct access for getting at the meaning behind decimal notation. To get at another mathematical construct, like fractions, it might be best to use pie charts or pattern blocks. Thus, using concrete referents can lead to a "direct and separate" treatment of mathematical constructs which, in turn, may lead students to develop a series of distinct and disconnected "pockets" of mathematical understandings. Even though the curriculum has traditionally spent some time trying to help students connect these constructs, we would argue that such efforts have often been too little and too late to help students fully develop a sense of the interconnectedness of various mathematical notions.

To construct a deep and rich understanding of decimal numbers, students need to understand the relationships among decimals, fractions, and percents. We believe the best way to help students accomplish this is not by encouraging them to build each of these notions separately, but rather, by helping them to first understand and explore the broad, underlying concept connecting these three constructs, namely ratio. In our view, students should first explore and understand the concept of ratio. They then can use that understanding as a tool to construct more specific instances of ratio such as fraction, decimal and percent. This is obviously a very different structure from traditional instructional sequences where students typically are encouraged to move from specific cases to more general understandings. However, we contend that if students first develop this type of broad-based mathematical foundation, they can more readily explore, connect and deeply understand a variety of distinct mathematical concepts, such as decimal notation.

Research Context

This teaching experiment took place over a three year period (1992-1995) with a heterogeneously grouped class of elementary school students (n = 20) in a public school of a small city. During a segment of each of the three years, the same group of students was taught math using a pilot curriculum built around the concept of "splitting" (Confrey, 1994). Splitting actions, which include sharing, folding, and magnifying, are believed to stem from primitive notions which occur intuitively in children. Since these notions can lead directly to multiplication, and concurrently to division and ratio, Confrey (1988, 1989, 1994) has argued that in order to support the intuitive splitting actions of children, students should be introduced to the constructs of multiplication, division, and ratio as a trio of mathematical ideas, early in their school careers.

Consequently, in third grade the students in this study were introduced to multiplication, division, and ratio simultaneously. When this same group of students was in fourth grade, instruction focused on strengthening the construct of ratio and introducing fractions as a subset of ratio (Confrey & Scarano, 1995). As fifth graders, having developed a rich network of mathematical ideas, students were introduced to decimals.

Students' introduction to decimals was done over a six-week period and was built around three open-ended, contextual problems which students worked on in small groups. The first of these problems, the Weights Problem, was intended to reacquaint students with the broader territory of ratio and proportion to prepare them for working with decimal notation. The second problem, the Decimal Olympics, formally introduced students to decimal notation through the use of metric units of length. The third problem, the Domino Problem, was a modeling problem. Students had to predict how long it would take a chain of 281,581 dominoes (the world's longest chain) to fall down. To investigate this problem, students used stopwatches which measured time to hundredths of a second, and thus had to operate with decimal numbers.

Prior to decimal instruction, students were given a written pretest consisting of items used in previous research studies (Hiebert & Wearne, 1985; Hiebert, Wearne & Taber, 1991; Wearne & Hiebert, 1988, 1989; Resnick et al., 1989). A parallel form of this pretest was administered at the end of decimal instruction. In addition to the written tests, all class sessions were videotaped, and a sample of four students participated in task interviews. The tapes of the class sessions and the interviews were reviewed and analyzed in an attempt to discover the ways in which students were developing their understandings of decimal numbers.

Quantitative Results[1]

Overall, the students in this study did exceptionally well on posttest measures. Again, these tasks were taken from published research studies on decimal notation. Since very few of the tasks were taught directly as a part of instruction, students were relatively unfamiliar with the types of tasks they were given on the written test. Even so, students showed significant improvement improving from a class average of 35.6% total correct responses on the pretest to a class average of 80.3% total correct responses on the posttest.

Qualitative Results

Given students' positive performance on the written tasks, the question then became: How were students able to construct such robust understandings of decimal notation? To answer this question, we looked to the tapes of class sessions and interviews. In viewing these tapes, it was clear that in making sense of decimal numbers, students appeared to use their understanding of ratio as a tool for constructing their notions about decimal notation. Although there were numerous instances in which this seemed to happen, we will present here three illuminating examples.

When first encountering operations with decimals, the students in this study tended to convert the decimal numbers to fractional or ratio form. In a task interview, one student used extensive ratio reasoning to convert the problem 1.3 + .24 to appropriate fractional form. She wrote: 1 3/10 + 24/100 = 1 3/10 + 2.4/10. (She eventually got an answer of 5.4/10, forgetting to add the whole number one.) Such a nontraditional fractional expression speaks to this student's grounding in the concept of ratio.

In more complicated problems, students used more complex ratio reasoning to ascertain what quantities a given decimal number represented. At the end of the Domino Problem, students had to present their estimate of the time it would take 281,581 dominoes to fall. All five of the small groups gave their time estimates in decimal form (i.e. 1.7 hours) but only three of the five groups gave these estimates in terms of hours and minutes as well. The two groups who only gave their time estimate in decimal form were asked by Confrey to give some meaning (in terms of hours and minutes) to their decimal estimate.

The first group dealt with this issue by estimating. Their prediction was 1.48 hours. They claimed that this was very close to 1.50 hours, and since .50 meant 50/100 and fifty is half of a hundred, then it would take approximately one and a half hours for all the dominoes to fall. While this was a fairly straightforward problem to solve, students did use ratio reasoning to explain their thinking.

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[1] For a complete discussion of these results, see Lachance (1996). Also see Lachance & Confrey (1995, 1996).

The second group struggled a bit more in their explanation. They estimated that it would take 1.7 hours for all the dominoes to fall. While they knew this meant one and seven tenths of an hour, they had difficulty figuring out how many minutes were in .7 of an hour. Finally, a student from another group offered assistance. "1/10 of 60 minutes is 6 minutes, then take 7 times 6 and get 42 minutes." In essence, this student used proportional reasoning to figure out this problem. And as soon as she explained her reasoning to this all-boy group, they readily understood her thinking. (In fact, two of the boys were a bit annoyed because they realized they should have been able to figure this out for themselves.) Both the boys' reactions and the elegance of the girl's explanation indicates that such ratio reasoning was familiar and well-understood by these students.

The third example also occurred during the Domino Problem. Students had to divide 281,581 by 80. Some students got an answer of 3519 R 61 while others got 3519.7625. The students were asked how the two answers to the same division problem were related.

> **Peter:** From my point of view, they're both the same answer. First of all you can see that the 3519 are the same. That's how many times the whole 80 goes into the number. But the other two are...in integral division, the remainder of 61, well 80 can't go into 61...well, a one whole 80 can't go into 61 without fractions. And...but in normal division, they used fractions or decimals, so the 61 eightieths would be equal to 7625/10,000.
> **Confrey** writes, " .7625 = 61/80" and says: Prove to me that these two are the same. How do you know it's reasonable that they could be the same?
> **David**: Well, they're both around 3/4.
> **Confrey**: Why?
> **David**: 75 is 3/4 of 100...divide 80 by 4 and you get 20 and if you times that by 3 you get 60.
> **Confrey** then goes through David's estimating procedure on the board and says: Now prove that they're equal. (No response.) Mike you know a way, you said it earlier.
> **Mike**: Divide?
> **Confrey**: Yes, divide. (The students use their calculators to divide 61 by 80 and cheer when they get an answer of .7625.)
> **Sam**: You could also simplify 7625/10,000 until you get 61/80. The simplify button on the calculator will let you do that.

In this discussion, students came up against a very difficult concept: reconciling the decimal form of a quotient with the remainder notation of the same quotient. It is clear that at least some of these students were relying on their experience with ratio and fraction to see the various connections between the two forms.

Conclusions

To students in this study, the ratio construct became an important tool for building a conceptual understanding of decimal numbers. Based on this experience, we believe that grounding decimal instruction in the territory of ratio allows students to build a much deeper and connected understanding of decimal numbers. In this study, students' understandings of decimals were enriched by their ability to recognize the ratio construct underlying both the fractional and decimal notation systems, and thus the inherent connection between these two symbol systems. We therefore argue that school mathematics curricula need to encourage students to develop broad constructs - like that of ratio - so that the students can then build understandings of the more specific notions - like decimals, fractions, and percents - embedded within and connected through larger conceptual territories.

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MODIFICATIONS OF CHILDREN'S MULTIPLICATIVE OPERATIONS WITH WHOLE NUMBERS AND FRACTIONS THAT MAY GENERATE RATIONAL NUMBERS OF ARITHMETIC[1]

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Nathan and Arthur, two children in our project on Children's Construction of Rational Numbers of Arithmetic, developed their operations for multiplying, dividing and simplifying fractions over a period of two years (grades 4 and 5). The two children worked in the context of specially developed computer microworlds with a teacher/researcher for 45 minutes a week for approximately 50 weeks (over the two- year period). Each of the 50 teaching episodes were videotaped. Through ongoing analyses of the videotaped episodes a model of the children's development of their multiplicative operations on fractions, and the modifications of their Generalized Number Sequence that may result in the Rational Numbers of Arithmetic are being constructed.

Children's Abstract Number Sequences

The basic hypothesis of our teaching experiment was that children would construct their fraction schemes through modification of their whole number operations based on their abstract number sequences (Steffe, 1992; Biddlecomb, 1994; Olive, 1994; Steffe & Tzur, 1994). Our models of Arthur's and Nathan's mathematical development at the beginning of this two year period indicated that they had both constructed what we refer to as a *Generalized Number Sequence*.

Steffe and Cobb (1988) developed the notion of children's abstract number sequences from their teaching experiments with young children. They described the development of three successive number sequences: the Initial Number Sequence (INS), the Tacitly Nested Number Sequence (TNS) and the Explicitly Nested Number Sequence (ENS). The following key psychological aspects of number sequences have emerged as a result of our work in the current teaching experiment:

- Number Sequences are mental constructs: schemes.
- The possible operations associated with a number sequence emerge from the interiorization of *activities* that children engage in through applications of their prior number sequence. That is to say that children may do things in action first what they are not yet able to do mentally.
- Re-interiorization is the product of *recursively* applying the operations of a scheme to the results of a scheme.
- An Iterable unit is the result of *reversible* operations (One iterated five times produces one five, that can be partitioned into five ones). *Reversible Operations* are produced through *recursive* thinking.

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[1] The research reported in this paper was conducted as part of the activities of the NSF Project No. RED-8954678, Children's Construction of the Rational Numbers of Arithmetic. All opinions expressed are solely those of the author.

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- The operations of lower order Number Sequences reappear in higher order sequences but are applied to the more complex unit items. This may be similar to Kieren and Pirie's (1991) notion of *Folding Back* in their *Recursive* model of mathematical thinking.

A Generalized Number Sequence (GNS).

The re-interiorization of the ENS results in *iterable composite units* in much the same way as the re-interiorization of the TNS resulted in iterable "ones." With an iterable composite unit of four, say, a child can conceive of "four" six times as being the same as having a composite unit of six with a *unit* of four items in each of the six unit items that constitute the six "ones." The "ones" in the composite unit of six have become place holders for any type of unit (singletons or composites). We propose that this re-interiorization takes place through the recursive applications of a child's units-coordinating operations to the results of those operations: abstract composite units. Children with a GNS can begin to build exponential structures. They can also coordinate at least three different levels of units.

Nathan's Strategy for Finding the Lowest Common Multiple

An example of the strategic power that a GNS provides can be seen in Nathan's construction of a procedure for finding the lowest common multiple (LCM) of two numbers. This construction took place during Nathan's third grade year during his work with the Fractions Project, one year prior to his work with Arthur. Nathan did not have as his goal the generation of the LCM of two numbers. The LCM was never referred to in any of our teaching episodes with Nathan. I use it only as a short hand name for Nathan's procedure.

Nathan's goal was to find a partition for a candy bar (in the Microworld TIMA: Bars, Olive & Steffe, 1994) that would allow him to pull out both one third of the bar and one fifth of the bar. His procedure was to count by three's and by five's until he found a common number in the two sequences. He would think to himself in the following way: 3, 6; 5, 10; 9, 12, 15; 15. It's 15. He would then put 15 parts in his bar, pull five out for one third and three out for one fifth. He was able to coordinate and compare his two sequences of multiples until a common multiple was found, but he also knew how many of each multiple he had used to get to this common multiple; he had kept track of each sequence. Nathan was able to carry out these coordinations because "three" and "five" were available to him as *iterable units*.

The following year Nathan was able to use his knowledge of common multiples to obtain a similar result in a more efficient way: In a more complicated task, situated in TIMA: Sticks (Olive, 1996), requiring the children to make a fraction of a stick starting from a different fraction (e.g. make a ninth of a unit stick using a twelfth of a unit stick), Nathan was eventually able to partition the 1/12 into three to make 36ths. He knew this

would work because "both 9 and 12 add up to 36...four nines are 36 and three twelve's are 36, so four of these will be 1/9." Nathan had begun a process that would lead to what we call a *co-measurement unit* for the two fractions.

Constructing Fractions of Fractions

The example given in the last section of making a fraction using a different fraction as a starting point was not easily achieved by the two boys (Arthur and Nathan). Even though we were convinced by their prior actions that both children had constructed a *Generalized Number Sequence*, it took Nathan four teaching episodes over a period of a month to construct a scheme that he knew he could use to solve such problems. For Arthur, the struggle took much longer. One stumbling block that they met was to name a fraction of a fraction as a new fraction. They could produce and anticipate a "fifth of a fifth" but did not know its value as a single fraction. In order to establish its value they had to iterate the result 25 times to reproduce the unit stick. They then knew that the result was 1/25. Their operations for making a fraction of a fraction, while recursive in nature, were not yet *reversible*. Eventually, they developed the ability to mentally project the partition of three equal parts of say one twelfth, into each of the 12 twelfths in the unit in order to establish the value of a third of a twelfth as 1/36.

Even after developing the ability to create and name a *unit* fraction of a *unit* fraction (as in the above example) the two boys had difficulties extending their strategies for finding, say 1/4 of 3/7 of a unit stick. One particular type of task, at first using the Sticks microworld and eventually transferring the task to the Bars microworld, proved to be engendering for this development. The different potential actions provided by both of these microworlds, we believe, were critical in the eventual development of a co-measurement scheme for fractions.

The task (given during the Spring semester of their fourth grade year) was to share *part* of a pizza (represented at first by a stick in the Sticks microworld) among a number of friends and to find out how much of a *whole* pizza each friend would get. For example: A pizza (stick) is cut into seven slices (pieces). Three friends each get one slice. A fourth friend joins them and they want to share their three slices equally among the four of them. How much of one whole pizza does each friend get? The typical approach that the two boys took to this type of problem was to partition the stick into seven parts, pull out three parts, partition each of those three parts into four parts and pull three of these parts out for the share of one friend (see Figure 1.) They would then iterate this share four times to check that it matched the part of the pizza (stick) that they were given.

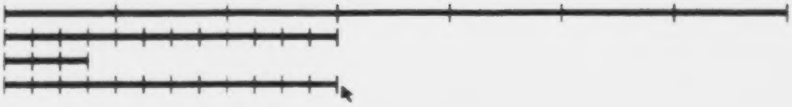

Figure 1: Four people share 3/7 of a pizza stick.

The difficulty for the two boys was in *naming* the share as a fraction of the original whole pizza. For instance, in the above situation Arthur named the share as ***three fourths of a seventh of the whole pizza***. He then attempted to *measure* the original stick with the share of one person by repeating it to make a stick the same length as the unit stick. The repeated stick, of course, came up short (see Figure 2).

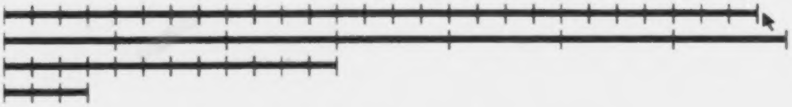

Figure 2: Nine iterations of 3/4 of 1/7 of a pizza stick.

This action was an application of Arthur's *Unit Fraction Scheme*, but this time the action was inappropriate as he was trying to iterate a non-unit fraction to reproduce the whole. The 3-part stick, that was one of the four children's share, was a co-measure for the three fours and the four threes that Arthur had created, but it was not also a co-measure of the whole 7-part stick. The goal of finding 3/4 of 1/7 of a stick was not attainable with his current operations that were based on his strategies for finding a *unit* fraction of a *unit* fraction--a modification of these unit fraction schemes was required that would enable Arthur to make units-coordinations with three different levels of units. He needed to be able to decompose the 3/4 of 1/7 as 3 of 1/4 of 1/7. He could then use his recursive partitioning operations to find 1/4 of 1/7, and use his uniting and unitizing operations to take 3 of these 1/28 as one thing. The constraints Arthur encountered in these episodes and the modifications he made in the application of his unit fraction schemes in overcoming these constraints provided the bases for the development of his *reversible partitioning operations*.

In the very next teaching episode with Arthur he appeared to make the necessary modifications in his schemes and developed a powerful strategy for finding the product of two fractions. The teacher had posed the situation of a pizza stick with nine slices, each slice having a different topping (indicated by filling each of the nine parts in a different color). She asked Arthur to pull out four different slices, which he did (three of them attached and one separate piece). The task was to share the four slices equally among five people *so that each person gets a piece of each slice.* Arthur divided each of the four slices into five parts and broke each slice up. He arranged the broken slices in five rows of four (one piece from each of the four slices in each row -- see Figure 3).

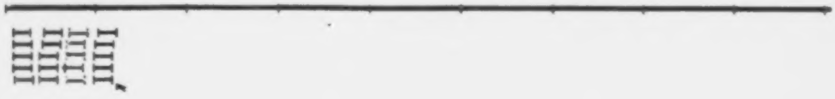

Figure 3: Sharing 4/9 of a pizza stick among 5 people

The teacher asked Arthur to join the pieces that make the share for one person and to find out how much this was of the whole pizza. Arthur joined the four pieces in one row and compared this share to 1/9 of the original stick. He then thought for more than one minute, looking intently at the screen:

From Teaching Episode on 3/10/93

T: *What do you think ?*

A: *I know it is 4/5 of a ninth of a pizza...*

The teacher confirmed his response and asked if there was any way to find out how much that was of the whole stick.

A: *Yes there is, but...* (trails off and thinks some more).

T: *How many of these small pieces do you have in the whole thing ?* (pointing to one of the four parts of one share)

A: *45*

T: *Why is that?*

Arthur explained: *There are nine pieces* (in the whole stick), *five in each, so that's 45.*

T: *How much of the whole pizza is one share then?*

A: *4/45.* He explains: *Because this is the share of one person..... and that's 4.... and in the whole thing there are 45, so the share of one person is 4/45!*

By focusing Arthur's attention on the unit of unit of units relation, the teacher helped to bring Arthur's *Recursive Partitioning Scheme* into play, enabling him to work out the unit fraction size for the smallest part. It was then a simple matter of uniting the four unit fraction pieces to establish the share as 4/45 of the whole. He was now able to decompose 4/5 of 1/9 as 4 of 1/5 of 1/9. These actions were the building blocks for Arthur's reversible operations with fraction quantities.

Fractions as Measures: A Path to Rational Numbers

While fractions had been constructed by Arthur and Nathan as both quantities and operations at this point in the teaching experiment, we did not think that they had constructed what modern mathematicians would characterize as "rational numbers." From our point of view a fraction scheme would need to include fractions as ***measurement units*** to be regarded as a scheme for generating the rational numbers of arithmetic. With fractions as measurement units, division of fractions becomes meaningful. For example, the questions "How much of 3/4 is 1/8?" or "How many eighths in 3/4?" require finding the *measure* of 3/4 in terms of 1/8 as a measurement unit. When children have constructed fractions as measurement units they have the possibility of constructing any fraction from any other by finding a *co-measurement unit* for the two fractions. From a mathematician's point of view it is just such a possibility that generates closure on the field of rational numbers.

All measurement involves the ratio comparison of two quantities. An indication that Arthur and Nathan had constructed fractions as measurement units would be their ability to make ratio comparisons among fractions. We developed several tasks in the TIMA microworlds that we believed would engender the development of this ability. One such task was modeled after the paper folding activities described by Kieren (1990) in which children compared the results of successive halving actions by folding a sheet of paper several times. We simulated a more generalized, successive fractioning activity using TIMA: Bars by successively partitioning and breaking a bar, and then continuing the process on just one of the resulting sub-bars. In one such task (one year after the prior teaching episodes) Nathan was to make thirds of the selected sub-bar on each of his turns and Arthur was to make fourths of the subsequent sub-bar. Both children were able to correctly identify the fraction name for the result of their actions by imagining carrying out their action on each of the *potential* sub-bars from the prior actions. We call this imagined activity "symbolic action." (Steffe and Olive, 1996)

The children continued taking turns until the resulting sub-bar was too tiny to go any further. Arthur correctly identified his tiny bar as 1/1728 of the original bar (see Figure 4). The teacher then asked the two boys to make comparisons among the pieces they had produced, such as "How many of those 1/1728 does it take to make 1/144?" The two boys were able to figure out the relation by thinking back through their respective actions. They were even able to figure out how many 1/144 it would take to make 1/6 of the unit bar, even though a 1/6 piece was not part of their sequence. They reasoned that it would take twelve 1/144 pieces to make 1/12, so it would take 24 to make 1/6 because "you have to double that to make 1/6."

The teacher then asked the boys to write down on paper the steps they took to make the 1/432 piece. Nathan wrote: *Unit ÷ 3 -> one third of that ÷ 4 -> one 12 ÷ 3 -> one 36th ÷ 4 -> one 1/144 ÷ 3 -> one 432 is the result.* Arthur wrote: *Took a unit bar, divided it by 3,*

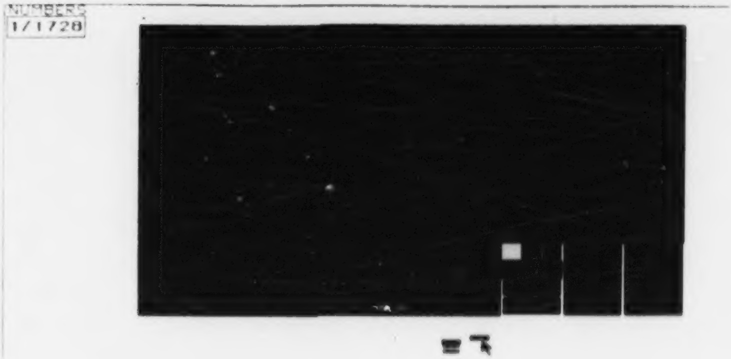

Figure 4: Measuring 1/1728 of a unit bar

took a third of that, divided it by 4, took a fourth of that divided it by 3, took a third of that divided it by 4, took a fourth divided by 3. The teacher then asked the children how their two methods were different. Nathan responded: *"He's using the new fraction as a measurement."* Nathan's last statement indicated to us that he understood the role of a fraction as a measurement unit. The teacher asked a clarifying question of both children at this point: *Do you mean Arthur changed the unit each time?* Both boys answered in the affirmative.

Subsequent activities in this same episode also indicated that both children realized that changing the order of their operations would produce some different fraction pieces but the end result would be the same if they took the same number of turns. They always represented their action in the microworld as whole number division and the result of their action as a fraction. In verifying that the resulting fraction would be the same if Arthur took all his turns as "fourther" before Nathan took his turns as "thirder" Arthur wrote down the following:

$$1\div4=1/4\div4=1/16\div4=1/64\div3=1/192\div3=1/576\div3=1/1728.$$

At this point the teacher asked Nathan if he could write the same thing using fractions and multiplication. He was not to use division. Nathan wrote the following:

$$1\times1/4=1/4\times1/4=1/16\times1/4=1/64\times1/3=1/192\times1/3=1/576\times1/3=1/1728.$$

He commented that this was just using the reciprocal: *"One third is the reciprocal of 3 and 1/4 is the reciprocal of 4. That's how I do division on paper."*

The above teaching episode indicated to us that Nathan and Arthur had constructed an isomorphism between the operation of taking a unit fraction of something and dividing by the whole number reciprocal of that fraction. Coupled with the notion of a fraction as a *measurement unit* these were strong indications that their fraction schemes were undergoing modifications that would result in a scheme for the rational numbers of arithmetic.

The modification of the children's fraction schemes to include whole number division as a reciprocal operation to fraction multiplication could also constitute a re-interiorization of their *Generalized Number Sequence* (GNS) *inwards* that produces recursive divisions of the number sequence rather than the outward direction of their GNS that provides for exponential structures. That is, instead of producing units of units of units, they could now produce units *within* units *within* a unit. And we believe that this re-interiorization *inward* is the necessary accommodation of a GNS that will generate *Rational Numbers of Arithmetic*.

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THE EMERGENCE OF MULTIPLICATIVE REASONING

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This paper reports on a second/third grade student's development of multiplicative thinking. Our analysis will focus on her constructions as she developed her thinking through the *anticipations* embedded in her activity. Our interactions indicate that engaging a student in activities which encourage imaging of arrays can encourage the development of multiplicative thinking. However, classroom tasks need to *not* assume that students are already imaging arrays but engage students in making such constructions.

Multiplicative Reasoning

In recent years there has been considerable research into multiplicative concepts at the elementary level (for example: Behr, Lesh, Post & Silver, 1983; Fischbein, Deri, Nello & Marino (1985); Nesher (1988); Vergnaud, 1988). However, much of this analysis has been formulated from adult perspectives of multiplicative structures rather than from children's ways of constructing multiplication. Using an alternative approach involving constructivist teaching experiments (Cobb & Steffe, 1983; Steffe, 1991), Steffe and his colleagues have investigated children's multiplicative schemes. Steffe maintains that the emergence of a concept of multiplication is possible once a child has constructed composite units, that is units that are themselves composed of units. Before this, children who have constructed the initial number sequence may learn to "unite sequences of counting acts into composite units" (1994, p.15). Their activity, however, does not suggest that they have constructed these situations in a multiplicative way; they cannot take composite units as given but "must make them in experiential contexts" (p.15).

Whereas Fischbein, Deri, Nello & Marino (1985) proposed repeated addition as students' primitive model for multiplication, Confrey (1994) suggests that approaches to multiplication that rely too exclusively on repeated addition "with an underlying basis in counting" (p.291) do not adequately explain what she has observed in the actions of children. Steffe (1994), in using children's number sequences as starting points, argues that: "For a situation to be established as multiplicative, it is necessary to at least coordinate two composite units in such a way that one of the composite units is distributed over the elements of the other composite unit" (p.19). He maintains that what a child is observed doing in counting, for example, the elements in three groups of four should not necessarily be interpreted as *only* repeated addition. The child's counting activity must be situated in what occurred before counting as the child developed a strategic plan of action. This plan may constitute a units coordination which can be interpreted as multiplicative. Thus, a child may *appear* to be using a strategy of repeated addition; however, the child's previous actions may indicate that this is merely a strategy for calculating a numeric value. The child may indeed have constructed a more complex multiplicative scheme.

Background to the Present Study

Clark & Kamii (1996) found evidence for some multiplicative thinking in 45% of second graders they interviewed. Yet only 48% of the fifth grade students they interviewed demonstrated "consistently solid multiplicative thinking" (p.41). Arrays are frequently used in teaching multiplication. For example, The National Council of Teachers of Mathematics, (1989) suggest: "Children can find the factors of a number using tiles or graph paper. This can lead to an investigation of numbers that have only two factors (prime numbers) and numbers with two equal factors (square numbers)." (p.42)

In investigating fourth and fifth grade students' constructions of multiplication in relation to their imaging of arrays we found evidence that students have not necessarily constructed an array mentally as an object with which to think. For example, Kristin, a mathematically gifted student, used a seemingly sophisticated partitioning procedure in an area situation but was unable to develop a justification for her method. She devised an alternative method by constructing an array to verify the correctness of her previous *answer* but was unable to use this thinking as her justification for the viability of her previous *method* (which could have been justified in this way) (Reynolds & Wheatley, in press). More in-depth probing of Kristin's constructions indicated further naiveté in her thinking in this setting. Another student, Tracy, was surprised to learn, as she 'measured' a rectangular region using a square template, that each row of squares she drew had the same number. Our experience with these and other students suggests that educators cannot make the assumption that students are already imaging (Reynolds, 1993) arrays when engaged in classroom activities which use materials such as tiles and graph paper to symbolize (Reynolds & Wheatley, 1994a) such multiplicative structures.

Setting

Because of our experiences with the fourth and fifth grade students the present research was designed to probe the multiplicative thinking of a younger student in a variety of settings, while focusing in particular on her constructions within an array setting. The research reported here encompasses three semesters of interactions with Sara, during her second-grade year and the Fall semester of her third-grade year. Interactions occurred weekly during the school year, each session lasting about one hour. We used a teaching experiment approach as elaborated by Cobb & Steffe (1983) and Steffe (1991). This approach enabled us as researchers to probe Sara's constructions and to pose tasks that might engage her in further elaboration of her multiplicative thinking. In interacting with one student in particular, we hoped to construct what Steffe (1996) refers to as "the epigenetic child," allowing us to suggest possible meaningful class activities to engage students in multiplicative thinking.

Much time was spent in exploring the structure of rectangles made with one inch square tiles. Sara was encouraged to explain her actions with the tiles and propose and explore questions and theories as she engaged in these activities. A variety of other tasks further extended her multiplicative thinking (for example, partitioning through the use of "sharing fairly" tasks and a variety of fractional tasks). In this paper we focus on Sara's constructions as she elaborated her imaging of an array. Our analysis will focus on Sara's thinking through the *anticipations* embedded in her activity: her reflection on her previous actions that led to her deliberate further investigation of events based on her a priori conjectures about those events.

Discussion

Initially,Sara showed little, if any, multiplicative thinking. When asked to share 16 packets of M & M's among three children who had each contributed $1, $1, and $2 to buy the candy her solution was a 2, 2, 12 sharing. Her reasoning was that the children who each contributed $1 thus had the same amount and the child who contributed $2 had "more." In another task she was asked how many fence posts would be needed to fence in a square, using 4 posts on a side. We have found in a variety of settings that students have difficulty coordinating "corner" posts in solving this task. This coordination requires that a corner post be simultaneously thought of as belonging to adjacent sides. Such thinking is needed if students are to think multiplicatively as described by Steffe (1994). Sara's solution was "sixteen" because there were four sides with four post on each (4 + 4 + 4 + 4). Initially she did not image posts in the corners; when she further thought about her solution she decided to add 4 more posts, one in each corner, giving 20 posts in all. However, Sara had a well developed sense of number sequences and had constructed composite units in her numeric thinking. Thus she had constructed numeric relationships which could be useful to her as we posed tasks that were designed to encourage multiplicative thinking (Steffe, 1994).

The following week Sara was given 24 tiles and asked to "make a rectangle." Thus began a series of sessions during which she explored rectangles made with various numbers of tiles and elaborated her imaging of an array. This elaboration enabled her to begin to construct multiplication. Our discussion of Sara's development centers around several key ideas. Although we will discuss each separately and in order as they appeared in her activity, each were intertwined and elaborated in interaction rather than in a linear fashion. Many of Sara's ideas were generated as she attempted to develop a strategy that would assure her of all possible arrangements with a particular number of tiles. Over the months she developed several ideas through this exploration.

Construction of Rows and Columns

Initially Sara constructed her rectangle as simply a collection of 24 squares. There was no evidence of her constructing rows and columns of tiles. However, when probed, she was able to investigate questions which centered around the number of tiles in a particular "row." Thus, one significant exploration concentrated on constructing units of tiles in "rows across and down" (how she explained her activity). Over several sessions she developed her thinking about her rectangles as composed of either m rows ("across") with n tiles in each row or n columns ("down") with m tiles in each, but not both simultaneously. Even though, when probed, Sara was able to examine rectangles she had made by focusing on the rows and columns, it was some time before she anticipated thinking about her arrangement in this manner. For example, one activity we used with her involved covering various sections of a rectangle she had made and asking her how many tiles were completely covered. She used one of two strategies: counting the number of tiles not covered and subtracting this from the whole (when she knew the total number of tiles in the arrangement), or using an estimation strategy informed by the estimated area of tiles covered. It was some time before she anticipated that she could use the arrangement of tiles in rows and columns as a thinking strategy.

1 by n Array

Previous experience with undergraduate students in mathematics methods courses had shown that many did not think of such an arrangement as an array. Interestingly, this arrangement was one of the early constructions Sara made. In the first session she made a 1 by 24 array. Two sessions later she posed the question: "Can I always make a one by whatever number rectangle?" She decided that this was always possible and subsequently assumed this arrangement when she was attempting to make all possible arrangements with a particular number of tiles.

Congruence of an m by n array and an n by m array

In the second session Sara made a rectangle with 6 tiles across and 4 down. Her second arrangement resulted in 4 tiles across and 6 down. Almost immediately Sara noticed that this arrangement could be turned 90 degrees to become the same arrangement that she had made previously. Over the next several weeks she frequently drew attention to this attribute using various arrangements. However, each was a noticing after the event. For example, in numerically listing all arrangements she had made she noticed she had "pairs" of numbers (3 across, 4 down; 4 across, 3 down). Eventually she constructed a relationship between these two "different" (her word) rectangles that enabled her to confidently use one arrangement in predicting its "pair."

An element is simultaneously a member of a row and a column

In the first session Sara attempted to predict how many tiles she would need if she added another row and column to an 8 by 3 arrangement. She decided 11 (8 + 3) and was surprised, when she subsequently added the tiles, that she needed 12. She could not explain why her addition strategy failed her here. Over the next several sessions it was apparent that each tile was considered as *either* a member of a row or of a column, but not both simultaneously. It was several weeks before we had any evidence that Sara was beginning to construct a way of thinking about both simultaneously. She had made a 4 by 7 arrangement which she was attempting to describe in terms of rows and columns when she said: "I'm wondering. Should it be six or seven?" She explained that since she had counted the top right corner tile in her "four across" she was unsure about counting it again in counting the tiles "down" the right side. She eventually decided that her rectangle should be described as 4 across and 7 down. Many times in subsequent interactions Sara was perturbed in her thinking about this relationship, but was not able to satisfactorily reach equilibrium.

In the last session the fence posts task was re-presented to her. This time in solving this task she decided to take a "bird's eye view" (looking down on the square). She placed two fence posts in each corner, one attached to each adjacent side. The construction of an element as simultaneously a member of a row and a column in an array is important if students are to think beyond a repeated addition model. Sara's struggles were an indication of how challenging such a way of thinking is. Reynolds & Wheatley (1994b) and Battista & Clements (1996) further evidence this aspect of imaging arrays. These findings support Steffe's (1994) contention that evidence for multiplicative thinking is provided by spontaneous coordination of composite units.

Implications

Sara constructed some significant ideas during this time. Her elaboration of an array structure could be used to enhance her multiplicative thinking meaningfully. Our analysis of Sara's activity suggests that we need to be cognizant of what Duckworth (1987) speaks of as "learning with breadth and depth" (p.70). Sara needed to explore and reflect on her activity in a variety of ways over time before she could anticipate the various structures she developed. This was particularly evident in Sara's perturbation with regard to an element being a member of both a row and a column. Our interactions indicate that engaging a student in activities which encourage imaging of arrays can encourage the development of multiplicative thinking. However, classroom tasks need to *not* assume that students are already imaging arrays but engage students in making such constructions. We are not asserting that imaging arrays is the only route to multiplicative thinking but it does seem to be a productive one.

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A COMPARATIVE STUDY OF TWO REMEDIATION STRATEGIESFOR PROPORTION PROBLEMS*

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For this exploratory study, two missing-value proportion curricula were developed with an emphasis on either a unit rate or a within scalar strategy. These different curricula were used with six adults seeking math remediation at a midwestern community college. Each adult participated in a 75-minute diagnostic interview, followed by three instructional sessions of the same duration. After use of either curricula, most adults incorporated aspects of instruction into their written problem solving, and all demonstrated a greater facility in organizing and computing missing-value proportions.

Many math textbooks outline the cross-multiplication algorithm (CMA) as the standard procedure for solving a missing-value proportion, perhaps because of its procedural consistency. However, current research in proportional reasoning dismisses CMA as a process devoid of meaning (Post, Behr and Lesh, 1988; Cramer, Post and Currier, 1993) and concludes that teaching the CMA in middle school is misguided (Lamon, 1993; Kaput and West, 1994; NCTM, 1994).

Our research group is developing mathematics software and curricula for adult remedial math students æ students who have likely been schooled in common math solution strategies, such as CMA. We have two primary design objectives. First, since successful computational solutions do not necessarily indicate proportional reasoning, we seek to develop computational methods that are connected to meaningful instruction; that is, each step in computation should have a semantic referent. Second, in the adult remediation programs that we target (community colleges, private industry and the military), available instructional time is often measured in hours and days rather than weeks or months.

With these factors in mind, the issues facing instruction of missing-value proportions are: Can instruction in CMA be avoided (or delayed)? What sorts of instruction might reasonably lead to conceptual understanding *and* solid computational performance? Although the latter question is further complicated by qualitative studies of middle school students that suggest that there is no "best" method for teaching ratio and proportion (Lamon, 1992), there are two methods that stand out as alternatives (or possibly complements) to CMA. We discuss these below.

Theoretical Framework

The methods of *scaling-up* and *scaling-down* have been found to be a natural extension of previously learned multiplicative strategies (Cramer, Post and Currier, 1989; Lamon, 1993; Kaput and West, 1994). Scaling refers to determining a multiplicative relationship

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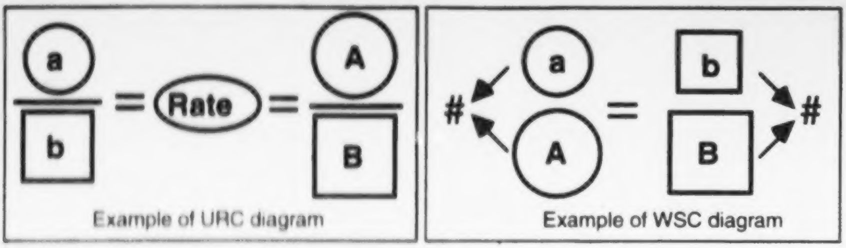
between two numbers—for instance, 3 increases to 12 by a factor of 4. When applying scaling strategies to missing-value proportion problems, students must determine the scalar relationship and use the "sameness of scalar operators to determine the missing term" (Lamon, 1993, p. 95). Scaling methods lend themselves to multiple representations in the form of linear equations, graphs, or ordered pairs, but scaling is limited in the scope of problems that can be solved successfully. Techniques of estimation or relative growth are required for problems with decimals or small differences between quantities. A *within scalar* strategy involves a multiplicative step which Kaput and West (1994) refer to as an abbreviated scaling-up/scaling-down process. Kaput and West (1994) suggest that the within scalar strategy is a fundamental informal strategy that may be a "direct cognitive descendant" of additive scaling strategies (p. 244).

When employing the *unit rate* strategy, a student first reduces two values from two different measure spaces to a unit fraction. For instance, 100 yards in 10 seconds is reduced to 10 yards/second. This new unit is then multiplied by or divided into a value from one of the two measure spaces. Post, Behr and Lesh (1988) regard the unit rate method as having the "most intuitive appeal," perhaps because the values throughout each step are meaningful. They surmise that children are familiar with unit rates (i.e., miles per hour, cents per food item, etc.) as early as third grade. The currency of unit rates in everyday communication may also make it an especially relevant method for adult learners that seek remediation in mathematics. Kaput and West (1994) believe that the unit rate method should be developed as a natural elaboration to division difficulties that arise with the previously mentioned strategies. Cramer, Post, and Currier (1993), in an analysis of the Rational Number Project, suggest that the unit-rate strategy must be emphasized before the introduction of CMA.

The Curricula

We wished to investigate how adults fare when exposed to remedial instruction we designed for missing-value proportion problems: a unit rate curriculum (URC) and a within scalar curriculum (WSC). At the core of each are diagrams that model semantic aspects of proportion problems. A key purpose of these diagrams is to bear some of the intellectual burden of proportional reasoning and problem-solving. Different shapes such as circles and squares signify difference in units, while size signifies difference in magnitude. For instance, with the WSC diagram, the layout is supposed to show that the results of the comparison between two like units equals that of the other two.

We designed worked examples (e.g. Ward and Sweller, 1990; Sweller and Cooper, 1985) to communicate these key concepts to students and to guide students in substituting problem elements for diagram elements as they solve missing-value proportion problems. Examples of diagrams used in the respective curricula are provided in Figure 1.



Example of URC diagram

Example of WSC diagram

Figure 1

Curricula were organized in modules of one worked example followed by two to four practice problems. In all, each curricula has 10 worked examples and 25 practice problems. The curricular materials are designed so that during the initial phase of instruction, practice problems are isomorphic to the previous worked examples. The final phase of instruction has practice problems that may be isomorphic to any of the worked examples, and, in a few cases, are not related to any of the worked examples.

Methodology

This exploratory study was completed at an urban midwestern community college serving a diverse student population. Six adults, each of whom were either currently working on or had completed self-paced, workbook-type instruction in fractions, volunteered for our study. The mean TABE math grade level for these students was 6.6.[1] Three students were female, three male; three were members of ethnic minorities. These students were tested using an instrument designed to measure proportional competence and elicit strategy use over a range of proportion problems. It contained fifteen missing value proportion problems similar to those used by Kaput and West (1994). During this session, adults were asked to verbalize their thinking as they solved each problem and were prompted by the interviewer if computations were enacted without stating supporting reasons. Based on interview notes, student work, and verbal protocols recorded via audio tape, students' proportional reasoning competence was assessed according to a scale adapted from work by Karplus, Pulos and Stage (1983) that identified student reasoning as Incorrect, Qualitative, Additive, or Proportional. Records were also kept regarding the degree of instruction provided to the student on each problem.

The six students were randomly assigned to work with either the URC or WSC curricular materials. Treatments consisted of three sessions of 75 minutes each, in which students worked together with a tutor. Performance on each problem was assessed on the basis of written work and evidence from verbal protocols, according to the same scale used in the diagnostic instrument. Five of the six students completed all sessions.

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[1] The Test of Adult Basic Education (TABE) was administered to students upon matriculation at the college, not immediately prior to this study.

Findings

All students demonstrated some qualitative, additive or proportional understanding when completing missing value proportion problems during the first diagnostic session, although three of the six students began the session with incomplete or illogical solution strategies, and all required some tutorial intervention. No students showed evidence of having mastered proportional concepts and solution strategies prior to instruction.

One student dropped out, but all remaining students demonstrated greater facility with missing-value proportions by the end of the sessions. A summary table of the data is provided in Table 1 below:

Student	Curric.	Initial Solution Strategy*	Initial Support Necessary	Final Solution Strategy	Final Support Necess.	Use of Tools from Curricula	Attempted to Routinize Method
JM (M)	WSC	A	Hi	P	Mid	Hi	Yes
AS (M)	WSC	P/A	Lo	P	Mid	Hi	Yes
JR (F)	WSC	I	Hi	P	Lo	Hi	Yes
FU (F)	URC	P	Mid	P	Lo	Lo	Yes
LM (F)	URC	I	Hi	-	-	-	-
MR (M)	URC	I	Hi	P	Hi	Hi	No

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* Solution Strategy Scale (Adapted from Karplus, Pulos & Stage, 1983)

I Incorrect or illogical

Q Qualitative comparison of given amounts

A Additive comparison strategy

P Proportional comparison strategy

Table 1

With respect to students and instruction, our data suggest the following:[2]

- With as little as four hours of instruction, participating students were able to meaningfully organize and compute missing value proportions without using the cross-multiplication algorithm (CMA).
- Early in the instructional sessions, students expressed a desire to understand why each computational step was necessary. In particular, one student (AS) was hesitant to use a calculator. He remarked, "The numbers won't make sense, I lose track of them with the calculator." This desire to "keep track of the number" within computation was expressed in different ways by all adults.
- Several students (JM, JR, LM, MR) showed a marked tendency to develop and then employ "algorithmic" or routinized solution strategies. That is to say, these students tended to recognize the problems as missing value proportion problems and apply routinized solution strategies to them. Once they could classify a problem they ceased to *reason* about it proportionally; rather, they operated on diagrams and symbols until reaching a solution.

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[2] Given the exploratory nature of this study, as well as its small sample size, we consider the data we gathered suggestive and all analysis provisional only.

- Students seldom referred to worked examples when solving similar problems.

- There seemed to be little relationship between students' level of proportional reasoning prior to instruction and their rate of gaining an improved understanding of proportionality. Whether or not a student would "take" to either strategy or method of instruction was unpredictable.

- Students tended to use the organizational tools in their own problem solving. Students very often drew similar iconic representations of the tools (see Figure 1) when problems became more difficult. Some students (such as JM) began all problems by drawing URC- or WSC-like diagrams.

With respect to the curricula, our data suggest the following:

- The URC thwarts to some degree students' efforts to routinize solution procedures. Although the first calculation step is always to divide the values of two different semantic units to form a unit rate (e.g., gallons into miles for mpg), the second step require either multiplication or division. Without sufficient experience with algebraic symbol manipulation, some relevant crystallized knowledge or experience canceling units, this second step can be difficult for adult students. Thus, URC is not procedurally stable. One student (MR), who professed a dislike for division, frequently failed on problems that required division in this second step because he simply chose to multiply.

- To a lesser degree, the WSC made routinizing solution procedures problematic. As with URC, students could treat the first calculation step in procedural fashion. If the student chose to divide the smaller value into the larger (or multiply up from the smaller to the bigger, as the case may be), a choice needed to be made about the next operation to be performed with this scalar relation. However, WSC (and its attendant diagrams) supports simpler computational procedures than URC.

- In both cases, however, proper use of the diagramming tools can help avert these difficulties. One student in the WSC condition (JM) sometimes could not decide whether to multiply or divide in this second calculation step, but seemed to be aided by the relative size of the sets in the diagram.

Discussion

This study suggests that a curriculum for the instruction of missing value proportions, based on the informal strategies of children, can be used successfully with adults, even though adults have disparate educational backgrounds. While there are some problematic features of both the URC and the WSC, because the calculation steps of each strategy carry meaning, they seem to us to be preferrable to CMA. Furthermore, we suspect that there are concepts and relationships imbedded in each strategy that can be used in a complementary fashion: (1) use of scaling or unit-rate methods alternately to *estimate* a reasonable result, (2) establishing a meaningful motive for calculating based on within or between measure space relationships, and (3) positioning quantities in ratios that elicit a quotient-type connection for calculation.

As discussed previously, most students adopted the representational forms used in the respective curricula. Also, most students wanted to routinize computational procedures. It is still unclear why the conceptual and computational understanding that these adults had prior to our instruction shifted to representations that were provided in our curricula, and why they seemed to desire the computational stability of an algorithm. These shifts were not immediate; they would often occur in the second or third session. Were these shifts initiated by the numerical complexity of the problems, the problem contexts, the instructional tools, or the amount of time invested in instruction and practice? These shifts may be artifacts of the population under study: adults, as self-directed, active learners (Tennant, 1988) may value computational stability more than conceptual understanding in the end. If these shifts are not restricted to adult populations, it suggests that curricula should be designed to provide computational stability as well as conceptual understanding. Our limited experience with six students suggests that this requirement may favor WSC over URC. Further study is needed to answer this question.

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THE INTERACTION OF INSTRUCTION AND CHILDREN'S CONCEPTIONS OF RATIONAL NUMBERS

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There are two related motivations for examining the interaction of instruction and children's understanding of rational number. The first is that "students create their own internal representations of the world and build their own networks of representations. A crucial aspect of students' constructive process is their inventiveness" (Hiebert and Carpenter, 1992, p.74). Even in school mathematics students rely on invented strategies to solve problems, but their inventions may not always lead to productive mathematics. "If students are working with written symbols unconnected to richer networks of knowledge, their inventions often produce flawed algorithms. . . . However, if the arguments of students' inventions are parts of well-connected networks, the resulting mathematics can be productive." (Hiebert and Carpenter, 1992, p.74). The second motivation is that "learning is characterized by the subjective reconstruction of societal means and models through negotiation of meaning in social interaction" (Bauersfeld, 1988, p. 39).

Thus, an environment with appropriate tasks and meaningful discourse needs to be provided so that children will be motivated to learn about rational numbers and will develop productive concepts of rational numbers. Teachers and their knowledge, as well as the curriculum materials they employ contribute to this environment.

A previous study will be examined from two points of view in order to provide a backdrop for discussion. The study involved specially developed curriculum materials and activities using manipulatives. The instruction was conducted by a classroom teacher who mainly used small group instruction allowing the children to discuss the curriculum materials prompts and how to apply the manipulative materials. Both points of view will focus on classroom interactions. The first on what children say in response to the prompts and what that indicates about their understanding. The second will focus on the teacher's guidance of the discourse and what that indicates about the teacher's goals and understanding and how it influences the classroom culture. When we use manipulatives and other contrived tasks, how do experiences with these tasks relate to children's conceptual knowledge of rational number? How do classroom interactions influence learning in classrooms and the quality of the concepts developed?

Bauersfeld, H., (1988). Interaction, construction and knowledge: alternative perspectives for mathematics education. In T. Cooney and D. Grouws (Eds.) *Effective mathematics teaching* (pp. 29-46). Reston, VA: National Council of Teachers of Mathematics and Lawrence Erlbaum Associates.

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TOWARD A THEORY OF CONTENT-SPECIFIC TEACHING: A CASE STUDY OF KNOWLEDGE USE ABOUT FRACTIONS AND RATES

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Objectives. This paper's objectives are two: (a) to examine in detail one teacher's knowledge of both mathematical and pedagogical representations about multiplication of fractions, rates, and the rules of the signs (FRS), and (b) to examine in detail the mathematical and pedagogical representations that the teacher uses for teaching FRS.

Participant and setting. At the time of the study, Mr. Kantor, the participant of the study, was teaching eighth grade algebra and had been teaching mathematics for about five years in a suburban school district known for high student achievement.

Conceptual framework. Teaching can be conceptualized as the process of constructing pedagogical events (explanations, representations, and questions) about content curriculum events. The purpose of these pedagogical events is to help students construct mathematical knowledge (Contreras, 1996). In the construction of these pedagogical events teachers may draw on their knowledge of both mathematical representations (e.g., definitions or symbolic representations and mathematical proofs) and pedagogical representations (e.g., pictorial representations and story-problem representations).

Data collection and procedures. There were three main sources of data: (a) videotapes of Mr. Kantor's teaching, (b) interviews and questionnaires, and (c) the textbook. I videotaped Mr. Kantor's instruction to capture in detail the representations that Mr. Kantor constructed for teaching FRS. Through a content analysis of the lesson *fractions and rates* I identified six content curriculum events (CCEs): the multiplication of fractions theorem, the rate model for multiplication, and the four rules of the signs. For almost each CCE, Mr. Kantor was asked to provide each of the four types of representations.

Data analysis and results. A content analysis of the data corpus revealed that Mr. Kantor's knowledge of representations was strong except in the case of mathematical proofs. Regarding knowledge use, he constructed a pictorial representation to illustrate why the multiplication of fractions holds using the numerical example $(3/4)(3/5) = 9/20$. He also used story-problem representations for teaching the rate model of multiplication. However, he did not teach any of the four rules of the signs.

Discussion. The discussion will focus on the potential of using the information about how one teacher taught FRS for (a) constructing theories of how teachers use their knowledge to teach specific mathematical content (FRS) and (b) rethinking what it means to teach for understanding a specific mathematical content (FRS).

Contreras, J. (1996). The teacher as a constructor of pedagogical events about content curriculum events: Toward an empirical validation of a theoretical framework integrating curriculum, teaching and learning in studies of learning to teach. Paper accepted for presentation at 18th PME NA.

INTERPRETING FRACTION MEANINGS: HEARING THE VOICES OF LANGUAGE MINORITY FOURTH-GRADE STUDENTS[1]

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The difficulties associated with learning fractions by mainstream students are well documented. The fast-changing demographics of the classroom calls for studies that build on previous findings indicating that cultural factors such as social practices and language do influence mathematics learning (Stigler & Baranes, 1988).

This study identified processes and themes through which six language minority students (Gr 4) revealed their knowledge about fractions during and after receiving ten hours of instruction on basic fractions using a curriculum based on Kieren's (1988) constructive model of knowing rational numbers. Three videotaped individual interviews were conducted over a one-month period. Each student solved eight fraction tasks and was asked to verify/support all answers using a model (continuous,discrete).

Findings revealed the bilingual students gained flexibility in partitioning a unit as a mechanism for fractions as parts of a whole. This scheme however, did not support fraction meaning in the case of students who persistently linked "thirds", "fourths" and "fifths" with whole numbers concepts. Images about fractions were limited to "halfs" and "shaded parts". Equality of parts was not consistently viewed as a constraint within the context of an area model. Furthermore, choice of fraction model seemed to have influenced students' language for articulating fraction meanings.

The analysis of videotaped interviews was guided by Kieren's (1988) conceptual triad identifying the experiences learners need to develop an intuitive knowledge of rational numbers. They include the assimilation of constructive mechanisms, imagery and language. Due to the limitations of this study, findings can not be generalized to other population.

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[1] Part of a dissertation directed by Dr. Vicky Kouba. The State University of New York-Albany, 1993.

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Eighteenth Annual Meeting

North American Chapter
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Volume 2: Discussion Groups, Research Papers, Short Oral Reports, and Poster Presentations (continued)

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History and Aims of the PME Group

PME came into existence at the Third International Congress on Mathematical Education (ICME 3) held in Karlsruhe, Germany, in 1976. It is affiliated with the International Commission for Mathematical Instruction.

The major goals of the International Group and of the North American Chapter (PME-NA) are:

1. To promote international contacts and the exchange of scientific information in the psychology of mathematics education.

2. To promote and stimulate interdisciplinary research in the aforesaid area with the cooperation of psychologists, mathematicians and mathematics teachers.

3. To further a deeper and better understanding of the psychological aspects of teaching and learning mathematics and the implications thereof.

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Preface

This program began with a meeting of interested volunteers on October 1995 in Columbus, OH during the PME-NA meeting. The results of the ideas discussed and suggestions made were taken to a meeting of the local program committee at The Florida State University where the themes of research cultures in North America, teaching and learning geometry, and social interactionism were selected. These themes became the focus of the three plenary sessions. Research cultures in North America are discussed in the three papers that were part of the plenary panel comprised of Fernando Hitt from Mexico, Carolyn Kieran from Canada, and Jeremy Kilpatrick from the United States. Learning and teaching geometry is addressed in the paper presented by Collete Laborde. Social interactionism as exemplified in learning environments is discussed by Heinrich Bauersfeld. The plenary panel, moderated by Norma Presmeg, ended with the audience discussing meta-questions raised through the three papers. No reaction papers were requested of the other two plenary papers in order to allow for participant reactions.

Included in the *Proceedings* are 75 research reports, eight discussion groups, 32 oral reports and 28 poster presentation entries. The one-page synopses of discussion groups, oral reports, and poster presentations are organized by topic along with the research reports following the pattern begun with the *Proceedings* of the 1994 PME-NA meeting. Proposers expressed first choice: 139 research reports, 20 oral presentations, 35 poster presentations, and eight discussion groups. Proposals for all categories were blind reviewed by three reviewers with expertise in the topic of submission. Cases of disagreement among reviewers were refereed by subcommittees of the Program Committee at The Florida State University. This process resulted in rejection or reassignment of about 43% of the research report proposals and about 37% overall.

Submissions for the *Proceedings* were made on disk, and produced by the ERIC/CSMEE staff. The format of the papers was adjusted to make them uniform. Papers are grouped in topic areas for the table of contents. An alphabetical list of addresses of authors is included in the appendix in Volume 2 with page numbers of their reports or synopses. In the case of multiple authors, submissions were made with presenting author(s) name(s) underlined.

The editors wish to express thanks to all those who submitted proposals, the reviewers, the 1996 Program Committee, the PME-NA Steering Committee for making the program an excellent contribution to ongoing research and discussions of psychology and mathematics education; Dean Jack Miller, College of Education, and the administration of the Department of Curriculum and Instruction and the Mathematics Education faculty at The Florida State University for their support; the graduate students for their endless work on the preparation of the conference and *Proceedings*; and the staff of the Gus Turnbull Center for Professional Development for the organization and registration of the conference.

Elizabeth Jakubowski
Dierdre Watkins
Harry Biske
October 1996

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SOCIAL AND CULTURAL FACTORS

MATH CORPS SUMMER CAMP: USING THE METAPHOR OF "FAMILY" DURING MATHEMATICS INTERVENTION WITH INNER-CITY YOUTH

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During a summer mathematics intervention program with inner-city, mostly African American youth, the metaphor of "family" and a team approach to instruction in which high school students tutored middle school students were used to establish a sense of community. Students' scores increased between a pre- and post-testing of their conceptual understanding of the mathematics addressed during the intervention. Due to the prominence of the extended family and a helping tradition in African American culture, a theoretical orientation toward such interventions which builds on these traditions is posited.

Introduction

The underrepresentation and underachievement of minority students in mathematics have been underresearched (Matthews, 1984; Reyes & Stanic, 1988). Consequently, the persistence of these phenomena, particularly among African American students, remains largely unexplained (Ajose, 1995).

Matthews (1984) suggests that one part of the problem with research on these phenomena may be its attention to students who are unsuccessful in mathematics. She posits a need for more studies that focus on minority students who are successful in mathematics or which draw comparisons between those who are successful in mathematics and those who are not.

In a plenary address at the Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Cobb and Yackel (1995) raise the troubling prospect that the beliefs and values which underpin current reform efforts in school mathematics education might well be in conflict with the beliefs and values of many inner-city teachers of mathematics. As a consequence, there is the very real "possibility that we will unknowingly foster even greater inequities" (p. 15) in the mathematics education of children than already exist.

Because beliefs and values are culturally situated and distributed, one response to the foregoing is to develop orientations to teaching which build on the cultural material which children bring with them to the classroom. For example, Au (1981) has previously demonstrated the efficacy of using the cultural tradition of Hawaiian children as an orientation to teaching them to read.

More generally, many writers have noted the difficulties which many African American students have in schools, particularly in urban areas (Hale, 1980; Hilliard, 1995; Morgan, 1980; Sleeter & Grant, 1991). Among the contributory factors which have been suggested is learning style. The preferred learning style of many African American students has a tendency to be more active (Morgan, 1980) and more people-oriented (Hale, 1980) than learning styles typically observed in students from white middle-class backgrounds. Matthews (1984) identifies a number of factors that have been shown to influence the learning of, and participation in, mathematics by minority students. While Matthews has chosen to separate these influences according to source (parent, student, or school), it is interesting to note a sociocultural thread which crosscuts her scheme: the communication style of the parents, the learning style of the student, and the instructional methods and student interactions of the school. An orientation toward teaching that is closely matched to such influences, as well as to an active learning style, has great potential for teaching populations in urban areas having proportionately large numbers of African American students.

Math Corps Summer Camp

In 1992 and 1993, the mathematics department at Wayne State University (WSU), in collaboration with the Detroit Public Schools (DPS), conducted Math Corps Summer Camps under the sponsorship of the Community Foundation of Southeast Michigan, the Herbert and Grace Dow Foundation, the Detroit Edison Foundation, and the administration of WSU. In 1995 the Math Corps Summer Camp was a collaboration of the mathematics department at WSU and mathematics educators from the WSU College of Education and was supported by a grant from the Detroit Urban Systemic Initiative (DUSI), which is funded by the National Science Foundation.

Summer Camp - 1992 and 1993

The 1992 and 1993 Summer Camps each included approximately 40 middle school students, 20 high school students, seven college students who were mathematics or mathematics education majors at WSU, one or two DPS teachers, and two WSU mathematics professors. The middle school students were divided into teams of 10, and each team was led by one of the college students, assisted by the high school students and the remaining college students. Each of these summer camps extended over a six-week period.

The participating middle school students were a mixed-ability group with respect to mathematics achievement, with grades in their previous mathematics course ranging from A through D. Regardless of students' past mathematical achievement, the underlying premise of Math Corps Summer Camp is to challenge students rather than remediate them. Thus, basic mathematical skills were presented and reinforced, but in the context of doing higher mathematics. The centerpiece of the middle school Summer Camp was a course on the real number system. This course introduced students to relationships between different kinds of numbers: positive and negative integers, rational numbers, decimal and other representations of real numbers, and irrational numbers. Instruction featured a Socratic approach in which students were encouraged to discuss questions posed by their instructors, so that they might come to their own understandings of mathematical concepts. The students also learned to use calculators as a problem solving tool. The other mathematical activities, which included work on advanced topics, group problem solving and mathematical games, were conducted within the team structure, under the leadership of the college and high school students.

The middle school portion of the Summer Camp was strictly a morning program. The afternoons were devoted to mathematics instruction on an advanced level for the 20 high school students. These students studied topics in discrete mathematics, probability and statistics, and introductory calculus, which were all presented at a college level. In addition, they were introduced to the graphing calculator as a tool for mathematical investigation and ended the day with group problem solving sessions.

While the intent of the entire program is for each age group to learn from those further along in their mathematical development, it is noteworthy that the learning was often two-way. The high school students deepened their own understanding of mathematics by working with the middle school students, the college students learned through their work with both the middle and high school students, and the professional mathematicians and mathematics educators associated with the project learned from the interactions of all three groups. Throughout both of these Summer Camps there was an emphasis on the team approach, and the staff attempted to foster a spirit of family.

In both 1992 and 1993, the middle school students who participated in the Math Corps Summer Camps were given a pre- and post-test covering skills and concepts inherent to the course on the real number system. The data from both years were analyzed using a matched-pairs design. The data from 1992 show a mean increase from pre- to post-test of about 1.79 standard deviations ($p<.001$), while the data from 1993 show a mean increase of about 3.33 standard deviations ($p<.001$). Looking at these gains in a different way, in both years the lower quartile on the post-test exceeded the upper quartile on the pre-test.

Summer Camp - 1995

The 1995 Summer Camp was somewhat larger than the two previous camps had been, involving 60 middle school students, 40 high school students, nine WSU mathematics or mathematics education majors, three DPS teachers, and three WSU professors: one from mathematics and two from mathematics education. As in previous Summer Camps, the students were organized into teams of 10 students, each led by one of the college students, assisted by six or seven of the high school students. Despite the larger size of the 1995 Summer Camp, we attempted to preserve those features of the earlier Summer Camps which we perceived to be successful. Thus, we based our work on four principles:

- *Math, math, math!* We conceive Math Corps Summer Camp as an immersion program in which students are surrounded by mathematical ideas throughout.
- *Remediation is best achieved through enrichment.* Math Corps Summer Camps challenge students at all levels to stretch their capacity to understand and learn mathematical concepts and master mathematical skills.

- *Kids teaching kids.* Math Corps Summer Camp is structured so that at every level students interact with other students who are a bit further along in their mathematical development. This leads to a second level of "remediation," because the act of teaching deepens the mathematical understanding of the "teachers" themselves.
- *Learning in groups.* Most of the mathematical activities incorporate group learning, open discussion, and students cooperating with one another in a problem solving setting. Moreover, the cooperative aspect is emphasized and strengthened by our team approach and our attempts to develop a family atmosphere.

The middle school students who participated in the 1995 Summer Camp were given a pre- and post-test similar to the ones administered in 1992 and 1993. The data were again analyzed using a matched-pairs design. The mean increase from pre- to post-test was 3.69 standard deviations ($p<.001$). Once more, the lower quartile on the post-test exceeded the upper quartile on the pre-test.

A Theoretical Orientation

One way to conceptualize the success of previous Math Corps Summer Camps is to view the team approach in terms of cooperative learning environments. Indeed, the power of cooperative learning has been widely reported (Bossert, 1988; Johnson & Johnson, 1985; Slavin, 1985; Yackel, Cobb, & Wood, 1991), and a synthesis of this research suggests that the effects hold for diverse populations across all content areas and ability groups. While we acknowledge the potential of cooperative learning environments when they are used with all students, we believe that the success of the team approach that inheres in our model for Math Corps Summer Camps runs deeper. In particular, because the overwhelming majority of our students are African American, we believe that the success of our team approach may have a cultural basis.

It is becoming generally understood that the extended family was a central, unifying feature of traditional African cultures, particularly those of West Africa, to which most African Americans can trace their ancestry (Gibson, 1980; Gutman, 1976; Mbiti, 1990; Myers, 1988; Zollar, 1985). So strong was the feeling of kinship and so broad the extended family in traditional African cultures that the individual, as such, did not exist except as part of a unified whole such as a family, clan, or "tribe" (Mbiti, 1990; Nobles, 1980). Martin and Martin (1985) note that a black helping tradition, which permeated nearly every phase of African life, is a natural extension of such extended notions of family. Many authors (Gibson, 1980; Gutman, 1976; Martin & Martin, 1985; Myers, 1988; Nobles, 1980; Zollar, 1985) suggest that, rather than eradicating such traditional features from African Americans' lives, slavery, the reconstruction period, and racial segregation in this country inadvertently supported the maintenance of these traditions.

Thus, if the extended family and helping traditions among African Americans have survived to the present day in urban areas, these traditions may well provide a theoretical basis for a more effective orientation to teaching African American children in the inner-city.

In particular, it is easy to conceptualize the teams in Math Corps Summer Camps as embodiments of a large extended family. In fact, by encouraging students to use the metaphor of "family" in thinking about the Summer Camp and their relationship to it, we clearly were suggesting that they view themselves as part of a unified whole. Moreover, the tutorial role played by the high school students seems an especially good fit with the notion of a black helping tradition.

Future Analyses

Once again, DUSI is funding a Math Corps Summer Camp during the summer of 1996. We expect this year's Summer Camp to mirror the previous ones, with one major addition. Based on our previous experiences with Summer Camps, we have developed hunches and tentative analyses regarding our successes into a theoretical orientation. We now need to ground that theory in data. Therefore, during the 1996 Summer Camp, the middle school students will be randomly assigned to teams, and one of the teams will be randomly selected for in-depth study. All of this team's mathematical activities during Summer Camp will be videotaped for later analysis.

By micro-analyzing the activities and interactions of one of the teams, we hope to begin to understand the success which Math Corps Summer Camps have demonstrated. In particular, we hope to tie our team approach and the metaphor of family to two aspects of African American culture: the extended family and a helping tradition. Doing so may inform the development of a model for mathematics intervention with inner city, and particularly African American, youth which might then be replicated elsewhere.

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DEVELOPMENTAL LEVELS IN CULTURALLY-DIFFERENT FINGER METHODS: ANGLO AND LATINO CHILDREN'S FINGER METHODS OF ADDITION

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English-speaking children in the United States follow a well-documented developmental sequence of addition solution procedures unsupported by teachers. This study contrasts that sequence with a different developmental sequence exhibited by Spanish-speaking urban children in which the fingers and number words function in ways opposite to that found in English-speaking children. Unlike the unitary English solutions, these Spanish solutions have the potential to support children's finger and mental solutions structured around ten; these methods are efficient and conceptually helpful for multidigit computation. Both kinds of methods were used in the same classroom. These methods came from homes in which the families had emigrated from Mexico, Central America, or South America.

The objectives of this paper are to describe cultural differences in early addition thinking for children from two cultures: English-speaking children in the United States (Anglo children) and Spanish-speaking children in the United States who have come from Mexico, Central America, or South America (Latino children). Anglo children follow a well-documented developmental sequence of addition solution procedures (see Fuson, 1992a, 1992b, for recent reviews of these methods). These methods are summarized and compared to a different developmental sequence of addition solution methods exhibited by Latino urban first-, second-, and third-grade children. This Latino sequence has more potential to support ten-structured methods than does the Anglo sequence. This paper exemplifies Vygotsky's position that cultural-historical conceptual tools affect thinking.

Methods

The new methods were identified in interviews of Latino children in a large midwestern urban school. Twelve second- and third-graders were given monthly interviews throughout the school year; four first graders were added to the study in mid-year to examine the earliest methods. Data were then obtained from two interviews with 50 Latino first and second graders and from classroom observations of these children and 150 of their classmates. Word problems and numeral problems were used. Half of these children were in math classes in which only English was spoken, and half were in classes in which only Spanish was spoken. Most children were from Spanish-background homes.

The research team carrying out this work was interdisciplinary (mathematics education, linguistics, psychology, anthropology, and classroom teaching), bilingual, and represented several different cultural backgrounds. The interviews were carried out by various members of the research team. Children's finger methods were recorded by writing letters for the fingers in a row as the child used them: T(humb), A (index finger), B, C, L(ittle finger), H(and). A dash was used to indicate fingers in between (e.g., T - C meant Thumb and fingers A, B, and C). The numbers the child said with each finger were written under the letter for the finger (usually only the first and last number for each addend needed to be written). A recounting of the fingers was written below the first count. A slash was made in the finger code at the end of the first addend. Fingers raised all at one time were circled (e.g., a circled H A - B meant that a hand and all three middle fingers of the other hand had been raised all at once). This method captured the features necessary to distinguish the two classes of methods and developmental progressions within them. It ignored orientation of fingers in space (children usually held fingers with palms facing them and looked at their fingers), which hand was first, and whether fingers were raised or lowered to count (usually fingers were raised). Interviewers were shown the typical methods and practiced recording them. Children who did an unanticipated method too rapidly for the interviewer to be sure of all details were asked to repeat their method. Reliability calculations carried out during training indicated that interviewers could be reliably trained to record methods accurately.

Results

Some children in each classroom used the Latino methods, and some used the Anglo methods. These numbers varied from class to class. More of the Latino methods appeared in the Spanish-speaking classes, but more children in all classes used the Anglo methods. The teachers of these children initially were unaware of the finger methods used by the children. Some children explicitly said that they had learned their finger methods at home or at school in another country. Others did not remember how they had learned.

Three developmental levels for each of these methods are shown in Table 1; the first level has two subparts: totals below and above ten. Because these methods are differentiated by how they use fingers and number words, we term them "word-total" (Anglo) or "finger-total" (Latino) methods because methods from other cultures can be classified in these more general ways. The two methods have three differences. First, for the second addend, the child either begins again from a starting finger so that the addends appear separately (Anglo), or the child continues counting across the fingers so that the second addend fingers are continguous to the first addend fingers (Latino). Second, the child counts the second addend by a finger pattern (Anglo) or by words (Latino). Third, the child hears the total in the words (Anglo) or sees it on the fingers (Latino). The first difference can be seen most clearly in the first two sections (in c and d). The second and third differences only appear in the third section (in b and c). Some children also did a combined method in which the fingers and words both counted on the second addend (Steinberg, 1984, also reported a few English-speaking children using this method). It was not clear how children were keeping track of the second addend, nor could they tell us.

Anglo and Latino children also show numbers on their fingers differently. Most children from both cultures begin with all fingers folded down and then raise fingers. Most Anglo children raise (unfold) the index finger, then raise the next three fingers in order (the long, ring, small fingers), and then raise the thumb. Most Latino children begin by raising the thumb and move successively across to the little finger, or they begin with the little finger and move successively across to the thumb. Some numbers are difficult to make with each method. The Anglo method skips around visually, so it can be difficult for another person to follow. These methods of showing numbers on fingers appear to be somewhat independent of the word-total/finger-total methods. Many more children showed numbers on fingers by beginning with the thumb or little finger than used the finger-total methods. Table 1 shows the most typical patterns: word-total methods beginning with the index finger and finger-total methods beginning with the thumb.

Our analysis of all of the methods used by our children identified three major levels of such use: counting all (2 sublevels), counting/adding on, and derived fact/ten-structured. Each level consists of an abbreviation or chunking of the previous level.
Both the word-total and finger-total methods begin by counting all: The child puts up fingers for one addend, then puts up fingers for the other addend, and then counts all of the fingers that are on were up. For addends ≤ five, word-total children show each addend on a separate hand, and finger-total children show the second addend by continuing with adjacent fingers. The word-total method shows each addend more clearly, and the finger-total method shows the total more clearly. As children learn finger patterns for particular numbers, they can use these finger patterns instead of needing to count out fingers when adding. Because the word-total method always uses the same fingers for each addend, word-total children can learn to put up finger patterns to show each addend. But they cannot use a single finger pattern to find the total; they must count all of the fingers or learn all of the different finger patterns that compose a number. The finger-total method always shows the total in the same form (e.g., seven is always a hand plus the next two fingers), so the total can be recognized as a single finger pattern. However, finger-total children must count the second addend or learn multiple patterns for the second added.

To count all for totals over ten, both methods require children to reuse fingers. For both methods, some children can do totals of ten and less but cannot do totals more than ten. Word-total methods of reusing fingers have not been widely reported in the literature, which concentrates on totals of ten or less. Our word-total children made the first addend on fingers, made the second addend by beginning again from the first finger, and then made each addend again counting them all (D. Geary, personal communication, January, 1993, also reported such methods by U.S. children in Missouri). Finger-total children continue on after the first addend, so they often only have to reuse the first hand (only for the three totals above 15 must they reuse the second hand).

In the next more advanced method (counting on/adding on), the intial counting of each addend disappears, and the counting of the first addend becomes abbreviated to saying the first word (even this is sometimes omitted). The counting of the second addend is embedded within the count of the total. The two methods diverge in the functions of the fingers and the number words. The word-total children use the fingers to keep track of the second addend, raising fingers until they reach a specified pattern while counting on from the first addend; the words say the total: 8 (first addend), 9, 10, 11, 12, 13, 14 while raising six fingers and stopping at the sixth finger (the words say the total 14). The finger-total children continue raising the fingers after those for the first addend, counting from 1 and stopping when they say the second addend number; the fingers show the total: raise 8 fingers, raise the ninth and tenth fingers while saying 1 and 2, reuse the first hand by raising successively four fingers while counting 3, 4, 5, 6 (the ten fingers and the new 4 make the total 14). Word-total counting on requires that the number-word sequence be at an automatized and abstract level for the words to be used to present the total, and it requires a cardinal-to-count shift in word meaning and a counting-word bridge to the second addend (Secada, Fuson, & Hall, 1983). The finger-total adding on is more concrete and more closely related to counting all: One still sees all of the fingers making the total. However, the word-total method is easily used for totals over ten because the fingers do not have to be reused, whereas finger-total children must reuse their fingers.

At the third level, numbers are chunked or recomposed to make the total. Word-total children use a variety of methods called derived facts (e.g., 6 + 7 = 12 + 1 because 6 + 6 = 12), but methods using ten are rarely used. Finger-total counting all and adding on show the second addend being separated into the amount to make ten and the amount over ten, and also show the total as ten plus the amount over ten. This ten-structured method is powerful and general and is conceptually and procedurally helpful for multidigit computation: The answer is already split into the ten that will get traded into the next column and the amount over ten that remains in the given column. Finger-total children do these methods with their fingers or mentally. Chinese, Japanese, and Korean children are taught to do this ten-structured method (Fuson, Stigler, & Bartsch, 1988; Fuson & Kwon, 1992). Many Korean first graders by the middle of the year could carry out such solutions mentally (Fuson & Kwon, 1992). This solution is made easier by the stated ten groupings in their number words (12 is "ten two", Fuson & Kwon, 1991/1992), and by the Korean finger method of folding fingers over ten back down again rather than reusing a hand.

When we began our research, many children were still using primitive count-all methods for finding single-digit totals in multidigit problems. They needed classroom support to move on to more conceptually advanced and efficient methods. We provided such supports in classroom teaching studies for both word-total and finger-total children, beginning with counting on/adding on and moving to ten-structured methods because of their utility in multidigit computation. We show in Table 1 ten-structured methods we have used for children following the word-total path. The second method is particularly easy for children. Steinberg (1984) reported one child who used a counting-on precursor of this method, but most word-total children in the United States seem to need classroom support to use any ten-structured method of addition.

Fingers are used as conceptual tools in many different cultures. Children will bring these methods into the classroom, providing opportunities for discussing different methods. If no finger-sum methods appear, a teacher could introduce methods described here as an alternative way to think about numbers around ten. Such discussions could meet both mathematical and cultural goals of introducing children to diverse points of view.

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A SOCIO-CULTURAL PERSPECTIVE OF SITUATED COGNITION IN THE MATHEMATICS CLASSROOM

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There is an implicit separation of situation and knowledge in the idea that two different situations can be used to teach the same thing, and in the idea that knowledge constructed in one situation can be used in a different situation. I present a model of the knowledge building process in which it is impossible to separate knowledge from the situation in which it is constructed, and yet the model can be used to explain how an experience in one situation might facilitate performance in another situation. The underlying assumption that makes this explanation possible is that problem solving does not involve finding a piece of knowledge that can be used in a new situation, but instead it involves interpreting and reinterpreting the new situation until it matches a previous experience. In this process, the new situation is transformed into an old one, and new knowledge is formed by the residue (Hiebert et al. 1996) left behind by the interpretation process.

Knowledge is constructed as we develop a coherent, conceptual organization of our experiences (von Glasersfeld, 1995), and so understanding how knowledge is constructed is a matter of understanding how experiences are organized and when they are organized. By experience here, I mean the situation and its surrounding context, or what Ableson (1979) called an episodic script. Ableson defined a script as a coherently linked chain of vignettes, where a vignette is the image and conceptual representation of an event of short duration. Both vignettes and scripts may be episodic, categorical, or hypothetical. (See Figure 1.) In this model experience is organized as it is used to form categories and hypothetical scripts. For example, a person might combine the vignette of counting blocks on a table and the vignette of answering questions about the number of blocks on the table to form a remembered sequence of experiences that constitutes an episodic script. At a higher level, a generic vignette of counting things may be combined with a generic vignette of answering questions about numbers to form a categorical script about counting. Hypothetical scripts are built from categorical scripts by abstracting and chaining critical features of categorical scripts together to form a series of possible scripts. For example, a young student may reason, "If I were to count these objects starting at this end I would get the same answer as if I started at the other end."

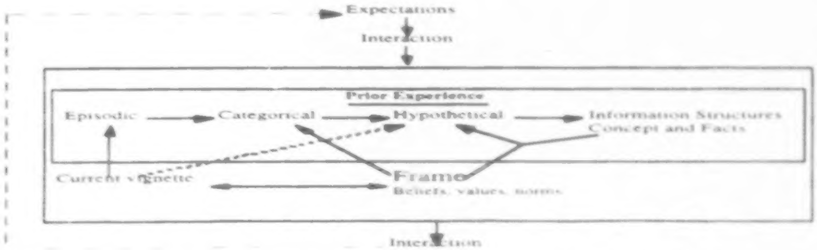

Figure 1 The role of frame in the cognitive processes that use prior knowledge and experiences to develop knowledge and to produce utterances.

Concepts and Facts. Hypothetical scripts become information structures, concepts, or facts through a process of internal and external validation (McNair, 1996). Validity is a product of the emotional, epistemological, and ontological residue that builds up as a result of our successes and failures to accomplish our goals in different situations. As we acquire more and more problem solving experiences in a particular domain and using a particular frame, the hypothetical scripts we create become a part of a coherent system of "concepts and actions that one has found to be successful, given the purposes one had in mind" (von Glasersfeld, 1995). Once a sufficient level of validity is obtained the hypothetical script is transformed into a concept or fact and becomes a feature of the associated categorical scripts. What this new concept, fact, or information structure means depends on the questions that confront the learner. For example when a child learning to count is given 17 pencils to pass out to her 21 classmates, and upon running out is asked, how many more do you need, the last digit in the counting sequence takes on an expanded meaning based on this new experience. It ceases to be simply an answer to the teacher's question, and becomes a useful tool that facilitates communication in a larger set of social contexts. As it does it may become a feature of a categorical script of classroom service experiences.

Frame

The mechanism that develops categorical scripts and that guides the construction of hypothetical scripts is frame (McNair, 1996). A frame is a set of socially constructed beliefs, values, and expectations that guide our thought and communication processes in a given situation (Goffman, 1975; Tannen, 1979). Frames are constructed in social situations and consist of the epistemological, ontological, and emotional residue we take from a problem solving situation. My hypothesizing about the rate at which a retirement fund may grow versus my hypothesizing about the derivative of a continuous function is guided by two different frames. However, the situation involving the retirement fund can be problematized in a way that makes hypothetical scripts, concepts, facts, and information structures about derivatives applicable. In this case, frames may be mixed and categorical features may be shared so that both derivatives and retirement funds acquire new meaning.

When a situation motivates a particular frame, that frame then guides all of our cognitive processes including the interpretation and production of utterances, the storing of memories and episodes, the categorization of episodes, the creation of hypothetical scripts, and the selection of appropriate information structures, concepts, and facts. When we encounter similar situations that motivate the same frame, then the referentially semantic content of the frame, which consist of all of the scripts, facts, concepts, and information structures that have been processed by that frame, becomes available in the form of things we know how to do in these kinds of situations.

Situation, Knowledge and Problem Solving

In the model described above, it is impossible to separate knowledge from the experience or the situation in which it was constructed. A situation is knowledge waiting to be constructed, and knowledge is a dynamic system of memories of past experiences organized by a frame which is itself constructed in and by the situation. Knowledge is a combination of a situation, an action, and a result. I can know what I did in a situation, and I can know the result of my action, but I cannot talk about or think about what I did without reference to the situation. I cannot use the knowledge constructed in one situation in a different situation, but I can do what I did if the same situation arises again. Frames are the socio-cultural tools we use to interpret new situations, and so my ability to solve novel problems is a function of my ability to use different frames to interpret the problem. It is my ability to interpret a novel problem as something I have done that allows past experience to facilitate problem solving, and so knowing is a matter of my ability to interpret and reinterpret a situation by using a variety of different frames.

According to the model presented here, problem solving is not a search for a usable piece of knowledge, it is a search for an understanding of the situation. In this search, the situation is transformed. In the problem solving process, I may use a number of different frames in an effort to see the situation in a way that allows me to know what to do. Each time I change frames I change my interpretation of the situation. For example, paying a restaurant bill and deciding what the tip should be is a situation that requires the use of many different frames. There may be a qualitative frame that determines the criterion on which the amount of the tip is based. There may be an ethical frame that determines the waiter's or waitress' right to a tip. There may be a financial frame that limits the amount of the tip, and there may be a calculation frame that calculates the amount of the tip. Each frame interprets the situation based on past experience. I do not use my knowledge of percents in this situation, but instead I define a percent situation which is the same as every other percent situation that I have experienced.

In this example we see that situations are made of smaller units that combine to form a whole. Each piece may be interpreted using a different frame, and so the situation itself is interpreted from many different perspectives. Each frame that is used in the situation is enhanced and changed by the other frames. Restaurants become places where percents can be used, percents become a factor in ethical decisions, and the knowledge that is the restaurant situation, or some smaller unit of it, might be used in an ethical discussion about rights and responsibilities.

Learning

Earlier, I defined frame as the epistemological, ontological, and emotional residue of our interactions with our environment and our peers. Hiebert et al. (1996) describe learning as a residue that remains after an activity is over, and they define three different kinds of residues. In this section, I interpret their definitions with respect to the model I have presented to demonstrate what the model has to say about learning.

The first kind of residue Hiebert et al. define is insights into the structure of the subject matter that are left behind as a result of analyzing patterns and relationships that may exist in a situation. In the model presented here, these insights are a recognition, or a definition, of a new feature of the categorical scripts to which the problem solving situation belongs. This was described earlier as the way hypothetical scripts become facts, concepts, or information structures by becoming a feature of the associated categorical script. Recognizing these structures requires the use of a frame that interprets across situations. This is a reflective frame used to understand the memory of our experiences, and it differs in some way from the frames we use to interact with our environment.

The second kind of residue Hiebert et al. discuss is strategies for solving problems, and they discuss two different kinds of strategies. First, they discuss the residue of particular procedures. In the model presented here, these are hypothetical scripts, concepts, facts, or information structures that have become features of a category. The procedure is the experience, or a hypothetical script that has become an abstract feature of a categorical script, so that using groups of ten becomes a strategy for counting when it becomes a feature of a group of experiences.

The second kind of strategic residue Hiebert et al. discuss is "general approaches or ways of thought that are needed to construct the procedures." This is what I refer to here as frame. Frames are a way of thought that produce a general problem solving approach. In the restaurant example, the interaction between frames produces one kind of strategic residue that might result from a problem solving situation. In this case a frame may become a feature of a categorical script. For example, if a mathematical frame becomes a feature of a categorical script based on banking episodes then the use of mathematics becomes a strategy for solving banking problems. It is the interaction between frames that allows us to develop new approaches and strategies.

Finally, and perhaps most importantly, Hiebert et al. suggest that students take new beliefs, values, and attitudes from a problem solving situation. These are the epistemological, ontological, and emotional residues that build up to develop the frame that plays a central role in the model defined in this paper.

Implications

A major implication of these arguments is that instead of trying to help students organize what they know, we should be helping them find ways to organize what they don't know, which is the new situation. We can help them do this by helping them learn how to ask questions that problematize (Hiebert et al, 1996) their environment. In this case, instruction would focus on helping students develop a number of frames that allow them to interpret a new situation from different perspectives, and emphasis on students'

ability to apply what they know would be replaced by an emphasis on their ability to understand and interpret new situations.

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PRESERVICE TEACHERS' CONCEPTIONS OF CULTURAL DIVERSITY

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Preservice Teachers' Conceptions of Cultural Diversity is a discussion group open to all individuals attending PME-NA 1996 that are interested in preparing teachers to teach in culturally diverse settings. This discussion will begin with a discussion of participants' definitions of culture and ideas for addressing diversity in mathematics classrooms. Next, preservice teachers' definitions of culture and ideas for addressing diversity in mathematics classrooms that were collected from students in methods courses for k-6 and 5-8 mathematics teachers will be presented. The preservice teachers' definitions of culture and ideas for addressing diversity in mathematics classrooms will be contrasted to academic definitions of culture and suggestions for addressing diversity. Lastly, the group will discuss programs that have been implemented to prepare teachers to teach mathematics in culturally diverse classrooms and the research that needs to be conducted to document the impact of these programs.

THE UNIVERSITY MATHEMATICS COMMUNITY AND MATHEMATICS EDUCATION: A QUALITATIVE STUDY OF CULTURAL FACTORS

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Cultural practices of the university mathematics community, most directly the tradition of lecturing in courses, conflict with many ideas proposed by the mathematics education community. Thus, mathematics faculty members' prevalent attitudes, beliefs, and behaviors toward teaching, learning, curriculum, and mathematics education must be considered along with educational research in the design of course reform efforts.

This paper broaches some of the themes emerging from a preliminary "constant comparative" analysis of data collected from three university mathematics departments in Louisiana. To date, data sources include mathematics and mathematics education publications, mathematics department "teaching seminars," casual conversations, and more than 35 interviews with instructors, classroom observers, graduate teaching assistants, professors, department chairs, and deans.

Mathematics faculty seem to be *generally* unaware of and uninterested in mathematics education, as indicated by my encounters with them and the continued prevalence of lecture courses, among other sources. In Krantz's (1993, p. 70) lecture oriented book on teaching mathematics, he remarks, "The last thing I want is for mathematicians to spend all day in the coffee room debating the latest pedagogical techniques promulgated by some Ivy League school of education."

Perhaps a slowly growing respect for mathematics education is reflected in two mathematics departments' support of a constructivism based college algebra course. Still, mathematics faculty do not embrace education as a theoretical domain. Rather, they approach it as a service discipline—a set of teaching methods. This is evidenced in "teaching seminars" and mathematics teaching publications which fail to sufficiently address *learning* (e.g., Krantz, 1993).

Reform minded mathematics faculty who have some grasp of mathematics education are minorities in their departments. They say that reform is a gradual process that will benefit from the retiring of some traditional faculty members.

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GOAL ORIENTATION IN SECONDARY MATHEMATICS

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In recent years, some of the psychologists studying academic motivation have suggested that student goal orientation is an effective tool for understanding student motivation and a viable framework for studying classroom practice as it relates to motivation. Given that motivation is often a problem in mathematics, particularly in lower level classes, this study was undertaken to test the hypothesis that students develop specific goal orientations in mathematics. Although several categorizations of students' academic goals have been suggested, this study focused on three goals that have significant implications for academic motivation. The first two are "performance goals" in which individuals try to maintain positive judgments of their ability and avoid negative judgments by proving, validating, and documenting their ability, and "learning goals" in which individuals work to increase their ability and master new tasks. The third goal is a "work avoidant goal," in which students simply do what they consider the minimum amount of work necessary to get by in a class.

To study goal orientation in secondary mathematics, a structured interview was used with 57 students from 4 different high schools. These students represented a variety of ability levels and were enrolled in mathematics courses ranging from general mathematics to calculus. This report focuses on students' self-reports about how hard they work and how much their work habits are influenced by a desire to master mathematics (learning orientation) as opposed to simply trying to look competent (performance orientation) or trying to get by with as little effort as possible (work avoidant orientation). Preliminary findings of the study include the fact that most students tend to have goal orientations that reflect aspects of both performance and learning orientations in mathematics. More specifically, except for highly work avoidant students, most seem to be concerned about how their teacher views them as mathematics learners but care very little about how they look in front of their peers. That is, students tended to be performance oriented with respect to their teachers but not their peers. Students showed learning orientation in that they generally wanted to learn mathematics but they wanted to learn it because they thought they would need it later on rather than because they found it interesting or enjoyable. Even the work avoidant students tended to think knowing mathematics was important, just not important enough to spend time on.

Students' teachers were asked to comment on the motivational levels of their students and specifically about what factors motivated each student. Teachers tended to think that grades are the key to motivating many students but they also saw peers as more influential than the students claimed that they were. As might be expected, both teachers and students saw a connection between effort and achievement.

GENDER DIFFERENCES IN EARLY STRATEGY USE: PARENT AND TEACHER CONTRIBUTIONS

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The purpose of this study was to examine how the development of gender differences in mathematics strategy use in the first grade is related to parents' and teachers' reported instructions. We were also interested in determining whether and how adult attitudes toward strategies are perceived by children. To do this, four children (2 girls, 2 boys) from 23 first grade classrooms participated in a longitudinal study. The children were interviewed individually outside of the classroom setting at three timepoints (September, January, and April) as they progressed through the first grade. These videotaped interviews yielded information about children's strategy use, specific strategy metacognitive knowledge, perceived parent and teacher attitudes about strategies, and children's attributional beliefs. Children's strategy use was categorized as being overt (use of manipulatives), covert (mental operations), or retrieval from memory. In January of the school year, teachers were interviewed about each child. The questionnaire inquired about the types of strategy and metacognitive instruction they taught each child. Similarly, parents were interviewed over the phone about the types of mathematics strategy training and metacognitive instruction in the home.

We first examined whether children's strategy use and metacognition at the beginning of the school year predicted the reports of instruction by parents and teachers in January. We found that children's strategy use and metacognition as measured in October did not affect parent instruction of strategies as measured in January. Children's early strategy use and gender, were more likely to have teachers who reported instructing metacognition in January. The correct use of retrieval and gender predicted teacher reported instruction of metacognition and retrieval strategies. Next, we examined the relationship between parent and teacher instruction and children's strategy use, metacognition, and attributions as measured in January and April. We found that parent instruction of overt strategies predicted the correct use of overt strategies for boys, but not girls. In contrast, the instruction of covert strategies by parents predicted the correct use of these strategies for girls. Parent strategy instruction did not predict the use of retrieval, children's metacognition or strategies. Teachers who taught their students to use retrieval strategies and who taught metacognitive information were more likely to have students who correctly used retrieval. Teacher instruction of retrieval strategies significantly predicted the correct use of retrieval for boys, but not for girls. Teachers who provided metacognitive instruction had students who were more likely to make ability attributions in January and April. In addition, teacher overt strategy instruction predicted ability attributions in January, but not April. We also examined the relationship between adult reported instruction and children's perceptions of parents' attitudes toward strategies. Parents who instructed their children to use overt strategies were *less* likely to have children who perceived parents to prefer "smart-looking" strategies. This was particularly true for boys. Similar results occurred for teachers.

TEACHER BELIEFS AND ATTITUDES

ACCESSING ONE TEACHER'S UNDERSTANDING OF THE TEACHING OF GEOMETRY THROUGH STIMULATED VIDEO RECALL

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This study examined one teacher's beliefs and practices about the teaching and learning of geometry. The examination was based on the hypothesis that a person's beliefs have influence on the way he or she conceptualizes and carries out a task. A teacher's beliefs about the nature of mathematics and of mathematics education may therefore affect the patterns of instruction and behavior of a teacher in the classroom. Data were collected through an interview with a teacher before a lesson, a video tape of his lesson and another interview that involved a discussion of episodes from the video tape. The findings of the study indicate that some beliefs are not consistent with practice. The context and constraints of teaching appeared to have restrained the teacher from matching his beliefs with practice.

Introduction

There is much interest in teacher beliefs and the relationship of these beliefs to instructional practices (Anders & Evans, 1994). This interest is based on the assumption that beliefs are good indicators of a person's decisions about life activities (Pajares, 1992). It has been argued that beliefs influence one's perception and judgment and also affect one's behavior at work. It is therefore assumed that, in order "to understand teaching from teachers' perspectives we have to understand the beliefs with which they define their work" (Nespor, 1987 p. 323) and how these beliefs match practice. Kagan (1992) concluded that "the more one reads studies of teacher beliefs, the more strongly one suspects that this piebald of personal knowledge lies at the very heart of teaching" (p. 85). The understanding of the relationship between teacher beliefs and practices is therefore critical to education.

Thompson (1992) has noted that the small amount of research on mathematics teachers' beliefs indicates that these beliefs, particularly teachers' conceptions of the nature and meaning of mathematics, tend to influence their instructional practices. Hersh (1986) wrote "one's conception of what mathematics is affects one's conception of how it should be presented. One's manner of presenting it is an indication of what one believes to be most essential in it" (p. 13). What teachers think about mathematics may therefore shape the kind of mathematical environment they create and thus the kinds of mathematical understandings that their students will develop (Hoffman, 1989 cited in Schoenfeld 1992). A study by Thompson (1984) indicates that "the observed consistency between the teachers' professed conceptions of mathematics and the manner in which they typically presented content strongly suggest that the teachers' views, beliefs, and preferences about mathematics do influence their instructional practices" (p. 125). This study investigated one teacher's understanding of the teaching of geometry. A teacher's understanding about the nature of mathematical knowledge in general and the learning of geometry in particular was explored as well as how this understanding plays out in his teaching of geometry.

Method

One of the main objectives of the study was to identify the goals or intentions for carrying out certain activities in class. The contention of the researcher was that an adequate description and interpretation of the teaching process must view teaching behavior in the context of a teacher's intentions. This involved gaining access to the thoughts and decision-making of the teacher. In a 'natural' Grade 10 Mathematics classroom, where part of the data was collected for this study, video-stimulated recall was deemed most appropriate to identify the teacher's thoughts and decision-making processes. The method involved the researcher first, video-taping a teacher's lesson and then watching the whole video tape with teacher to enable him to 'relive' the lesson. Some episodes from the tape were then replayed and the teacher in retrospect gave a verbalized account of his original thoughts. Prior to collecting the data for this study, a couple of teacher's lessons were video taped and episodes were played and he was asked to recall certain events and his thinking process. This was done to establish rapport with the teacher and to ensure that the teacher was familiar with being video taped and the recall process. The video-stimulated recall also assisted both researcher and teacher to reconstruct and interpret certain teaching events. Before the lessons were video-taped, the teacher was interviewed about his beliefs on mathematical teaching and learning especially when it involved the teaching and learning of geometry.

Data/Analysis

In the pre-lesson interview, the teacher indicated that geometry in Grade 10 involved "proofs, logical reasoning, and arguing." He said that one of his goals when teaching geometry was to help students to think logically and defend their ideas. He also viewed geometric learning as being "more visual" than other topics like algebra and suggested that it thus requires concrete materials and diagrams to help children form imagery of geometrical concepts and processes. He claimed that for children to take "ownership" of their mathematical learning, they need to "visualize" mathematical concepts and processes in their minds, "talk to themselves" about mathematics, discuss mathematics with friends, do mathematical exercises on their own, and vocalize mathematical concepts and thinking processes in appropriate mathematical language. He saw mathematical understanding as involving "learning a whole new language". "I feel like [I am] teaching a new language" he said. As much as he would like his students to use their own words to express their mathematical ideas, he always encouraged them to use the conventional mathematical language. He saw mathematics as "made up of many components with its own language." He compared mathematics learning to learning how to ride a bicycle. He suggested that children need to acquire the habit and dispositions of interpretation and sense making.

Some features of the teacher's actual teaching included: insisting on students using the appropriate mathematical vocabulary, requiring students to write their solution to problems on the board and using these solutions for discussion, emphasizing the importance of both process and final answer, and relating a geometrical process to an algebraic process. In solving problems involving proof, he always advised students to establish semantic connections to diagrams and together with the "given" conditions, to draw their conclusions.

In the post-lesson interview, the teacher acknowledged his belief in maintaining order and discipline to allow him "to take care of business." Asked to comment about his lesson after watching the whole video tape, he said "I am happy about the lesson because students cooperated well while I took care of basic stuff."

As evidenced from the regular assignments which the teacher gives students after each lesson, it appears that he favors the idea of students engaging in mathematical activities both inside and outside the classroom. The teacher consistently gave students homework from their workbook and went through these assignments at the beginning of each class. The teacher believed that the discussion of students' solutions of problems on the board helped him to assess the general understanding of the whole class as well as the understanding of the individual students who contributed in the class discussion. The assignments, according to the teacher, gave the students the opportunity to discuss mathematics among themselves.

The teacher, who also teaches grade 12 mathematics, suggested that because there is so much content to be covered in grade 12 mathematics before the final examinations, he is compelled sometimes to teach "to the grade final goal" which is the final examination. "While I try not to be totally dictated to by the exams, it plays a big part on the decision on what I teach" he said.

The data seem to indicate that the teacher believed the mathematical community has its own practices and language and that mathematical learning is primarily a function of practice and communication. In his teaching practices, the teacher had the tendency to encourage students to use the appropriate mathematical language in the class. Asked about why he insists on students using the 'correct' mathematical terms the teacher responded "the main point is to communicate ... to help students to speak the same language."

The teacher's teaching practices seemed not to be consistently influenced by his understanding of mathematical learning. Geometrical learning, according to the teacher, required students to form imagery of concepts and processes through manipulation of concrete objects, diagrams, and appropriate vocabulary for geometrical concepts. However, the teacher neither used physical objects nor made any connection between the geometrical concept(s) and concrete materials during the lessons that were observed.

Analysis of the post-lesson interview indicated that the teacher used concrete objects only when he felt students did not understand concepts through diagrams. He claimed that "grade 10 geometry which is mostly 2-column proofs did not lend itself to three dimensional objects." It was also observed that while the teacher desired that the students talked to themselves about mathematics, he did not encourage them to work in groups in the class.

Conclusion

By exploring the intentions of a teacher, through a video-stimulated recall, the study has provided an insight into how and why some epistemological beliefs may not match actual teaching practices . The teacher's beliefs played a major role in defining teaching tasks and organizing the knowledge and information relevant to those tasks; however, it is quite clear that the context and constraints within which the teacher worked determined the consistency or inconsistency between the teacher's beliefs and practices. While the teacher seemed to believe that the construction of mathematical understanding involved a process of socialization of students into a mathematical community of practice, the desire to cover more content hindered the teacher from creating a classroom consistent with this belief.

These findings are not surprising. A study with similar objective conducted by Hoffman and Kugle (1982) was unable to find a "strong relationship between teacher beliefs and teacher behaviors" (p. 6) in the classroom. In a case study of 10 elementary teachers, O'Brien and Norton (1991, cited in Anders & Evans 1994) acknowledge the complexity of teachers' pedagogical beliefs and concluded that the availability or non availability of teaching materials and expectation from administrators and peers constrained the teachers' instructional decisions.

The findings have implications for teaching and teacher education. Most obviously, it suggests that in order to understand why teachers organize and run classrooms as they do, we need to pay much more attention to the goals they pursue and the constraints within which they perform their duties as teachers.

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EXAMINING TEACHERS' EVOLVING BELIEFS AND PRACTICES WITHIN A DIALOGIC COMMUNITY

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The study described in this report examined the complex interplay among teachers' beliefs, mathematics instruction, sociomathematical norms, sociocultural norms, and the evolution of dialogic community among four elementary teachers. This report addresses the relationship between the dialogic community and individual teacher's classroom practices. Findings suggest the dialogic community greatly facilitated the teachers' abilities to critically reflect on and make changes in their classroom practices and beliefs about mathematics instruction.

There is an increasing body of literature on teachers' beliefs (Pajares, 1992) and reflections (Ross & Hannay, 1986; Schon, 1987; Smyth, 1992; Sparks-Langer, 1992). A powerful yet understudied aspect of teaching practices and teacher change, however, is the influence the school community has on teachers' activities in their classrooms. How may teachers' beliefs be shared and influenced by a dialogic community? How may those shared beliefs and personal visions encourage teachers' reflections for collective action and possible change?

This report is based on a study which examines the complex interplay among teacher beliefs, mathematics instruction and the evolution of a dialogic community among four elementary teachers. The focus of this report will be to answer the following question:

Question: What is the relationship between a dialogic community and individual teacher's classroom practices with respect to mathematics instruction, sociocultural norms, and sociomathematical norms?

Theoretical Bases for the Study

This study is grounded in systems theory of organizational dynamics and constructivist perspectives of meaning-making. Both of these perspectives are important for understanding and examining teaching practices, beliefs, and underlying world views implicit in actions, stated beliefs, and classroom dynamics.

Systems theory tells us that an open system must be understood through the process of its interaction within its environment. The system cannot be studied in pieces. Underlying perceptions of order are complex interactions which often are not discernible. Organization within complex systems is governed by mutual connections and relationships of systems rather than hierarchies and simple causal connections. Classroom dynamics in interpersonal relationships represent open systems whereby complex interactions and mutual connections prevent attribution of simple causes to classroom events.

Systems theory is consistent with constructivist perspectives. According to some versions of constructivism, reality is constructed individually as well as socially (Cobb et al., 1990; Yackel & Cobb, 1995). Reality is multiple, intangible mental constructions. Therefore, it cannot be predicted or controlled (Lincoln & Guba, 1994) and cannot be studied in isolation.

Study Design

The primary researcher worked intensively with the group of four teacher-participants over an 18 week period. Informal, on-going relationships before and after the study contributed to the trustworthiness of the data collection process (Lincoln & Guba, 1985) and the establishment of a dialogic community which included the primary researcher as an active participant and facilitator. All four teacher-participants were Caucasian and had teaching experience ranging from two to twenty five years.

The data collection process occurred in two phases. The first phase of the study began August 16, 1995 and ended August 30, 1995. The purpose of this phase of the study was to gather background data and included the following components: (1) group discussion and negotiation of the procedures of the study with the participants, (2) individual participants completion of questionnaires, (3) three two-hour group meetings for further collection of background information, and (4) observations of participating teachers' classrooms starting the first week of school. The background data was key to inferring changes in practices and beliefs as the study progressed.

The second phase of the data collection process started on September 1, 1995 and ended January 30, 1996. During the second phase, the primary researcher (1) observed each teacher during her mathematics instruction once every week, (2) immediately followed each classroom observation with one-on-one interviews, and (3) met with all participants every two weeks for approximately two hours to discuss teachers' reflections on their classroom activities and mathematics dialogue topics. The topics for dialogue were suggested by participating teachers and the primary researcher one week prior to the group meeting and were distributed in advance. All classroom observations, one-on-one interviews, and group dialogue meetings were audiotaped and transcribed for the teachers to review before each group meeting. The intensity of the interactions among the participants during the second phase of the data collection process contributed to the establishment of a dialogic community.

Data from transcriptions of classroom observations, one-on-one interviews, and mathematics dialogue meetings, as well as field notes, and teachers' written responses to questionnaires were analyzed using a constant comparative method (Lincoln & Guba, 1985). The emerging categories from the multiple data sources were examined using a matrix of categories for comparing mathematics instruction, sociomathematical norms, and sociocultural norms within the classroom setting as well as changes within the dialogic community.

Findings

Findings of this study suggest the dialogic community greatly facilitated the teachers' abilities to critically reflect on and make changes in their classroom practices and beliefs about mathematics instruction. Sociomathematical norms, the ways teachers engaged students in and negotiated with their students meaningful mathematical activity, seemed to be influenced by heightened awareness and critical reflection of participants within the dialogic community. Critical discussions of classroom practices and assumptions about mathematics learning seemed to be necessary for and was influential in helping teachers approach their mathematics instruction differently. Discussions critically examining the importance of having children verbalize their mathematical thinking and provide explanations and clarifications of their mathematical problem solving seemed to influence and affect teachers' approaches to mathematics instruction. The establishment of a dialogic community similarly seemed to rejuvenate and energize the teachers in ways that allowed them to critically examine their own professionalism and relationships with other teachers. Especially important to the findings of this study was the establishment of the supportive and synergistic environment of the dialogic community.

As the study progressed, some significant changes occurred in all the participating teachers' classroom practices and beliefs. Weeks nine and ten were bifurcation points for these changes and correspond with significant dialogic discussions suggesting the dialogic community had an impact on all the participating teachers. For example, Macy's and Julie's instructional strategies shifted significantly from teacher centered to student centered. Small group cooperative learning, utilizing strategies such as round table and think-pair-share, became the main instructional mode. The sociomathematical climate in both classes became more open with increased emphasis on problem solving and probing questions. The students were encouraged to explain and justify their solutions within their small groups. Macy's and Julie's roles changed significantly from controlling the classroom practices to facilitating children's learning. As the study progressed, these teachers placed more emphasis in their classrooms on listening, observing, and asking questions. In addition, their beliefs regarding teaching and learning significantly changed. Macy realized she put limits on her children. By observing, listening, and asking questions she came to believe that a lot of times those children who are not labeled "gifted" can do mathematics problem solving beyond what she expected of them. Julie realized she needed to free her children to "fly" and to believe that they can "fly." By critically examining her teaching and changing her classroom practices and routines, she came to question the belief that mathematics learning occurs sequentially and she placed greater value on student autonomy in learning.

While Macy's and Julie's beliefs and classroom climates in the form of sociomathematical norms accompanied changes in instructional practices, Lucinda critically challenged existing beliefs even though her classroom practices followed a similar pattern throughout the study. By reflecting on her teaching, Lucinda became aware of the tension she felt between the "objective," evaluative part of teaching and the "subjective" level emphasizing student struggles to make learning meaningful. By thinking and trying to reason why she separates assessing learning from enhancing learning, she realized the problematic nature of her process/product dichotomy. She called it the "sad" part of her teaching and she continues to struggle with this duality or tension in her teaching.

Elizabeth's instructional practices changed from week nine when she integrated small group cooperative learning in her mathematics instruction. After 20 years of teaching, this year, for the first time, Elizabeth integrated small group interactions in her instructional strategies. She saw the value of children's collaborations in mathematical problem solving situations. Her classroom sociomathematical norms were dominated by valuing children's risk-taking, diverse thinking, and multiple explanations in the first nine week period. As the study evolved, particularly in the second period starting with week nine, she placed greater value on students' collaborative efforts for problem solving. For this change, Elizabeth gave credit to her reading current literature provided during dialogue meetings and to the influence of the dialogic community on her classroom practices. Elizabeth's beliefs regarding the value of small group cooperative learning as a source for creating learning opportunity for her children significantly changed.

Conclusions

It is very difficult to pinpoint the cause of these changes in each participating teachers' instructional practices. It is not possible to explain these changes by cause and effect paradigms. The appearance of change was mutually connected to the contribution of each participating teacher as the study evolved. It seemed the synergism of common vision among these four teachers in the dialogic community played a significant role for these changes to occur. Their common vision created a powerful energy that pulled the teachers together as a caring community, empowering them to emancipate themselves and critically examine who they are. This critical reflection and action (praxis) was a result of the dialogic community and participation in the research project.

As it becomes more and more important for teachers to negotiate with their students and with each other the important mathematics to be studied, teachers must learn to establish dialogic communities. The creation of dialogic communities may require drastic shifts in existing school structures. As this study suggests, the intensity of the experience for these teachers would be difficult to establish in most traditional school settings. The creation of a true learning organization (Senge, 1990) and the establishment of dialogic communities may be the cornerstones of tomorrow's schools.

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THE DEVELOPMENT OF PRESERVICE ELEMENTARY TEACHERS' BELIEFS ABOUT AND KNOWLEDGE OF ALTERNATIVE MATHEMATICS ASSESSMENT

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This paper reports a subset of findings from an extensive investigation of 61 preservice elementary teachers' beliefs about and knowledge of alternative mathematics assessment. These preservice teachers, who enrolled in a mathematics methods course, responded individually to four mathematics assessment surveys, participated in three paired mathematics assessment activities, and developed an alternative assessment project. This study explored a wide range of issues related to mathematics assessment. Findings reported herein include those related to (a) beliefs about mathematics assessment, (b) the extent to which various alternative assessment techniques can provide important information about students' mathematical learning, and (c) the role of Assessment Standards (NCTM, 1995) in mathematics assessment practice.

A strong movement for change in mathematics assessment has emerged (Kulm, 1990; National Council of Teachers of Mathematics [NCTM], 1995; Webb, 1993). Broadly viewed as a tool used by teachers to help students achieve educational goals (Webb, 1993), assessment in mathematics is no longer a matter of counting right and wrong answers on tests. Because mathematics has historically been viewed as "readily amenable to breaking into nice, simple, linear pieces" (Kulm, 1990, p. 1), assessment of mathematics learning has taken the form of measuring whether or not specific behavioral objectives have been met. As a result, anything that could not be stated and measured behaviorally gradually disappeared from the curriculum. Today, as the mathematics curriculum projects a new philosophy of a dynamic mathematics curriculum, assessment of the learning of that curriculum has also become more dynamic. Consequently, as mathematics assessment practices change, it is imperative to consider how assessment must be seen as an integral part of instruction (Cooney, et al., 1993).

Assessment practices implemented by teachers send a powerful message to students about what types of mathematical thinking and mathematics content are valued. What gets assessed and how it gets assessed implicitly send signals to students about what teachers believe is important. Most often, teachers have not considered how their beliefs affect their teaching and assessment practices, nor have they reflected on how changes practice may run counter to students' expectations of mathematics practice (Borasi, 1990) and, as a result, affect their students' beliefs.

Many alternative assessment techniques have surfaced, including formal and informal student self-assessment, and assessment of mathematical problem solving (Moon & Schulman, 1995). These techniques offer both advantages and disadvantages in terms of time commitment, ease of implementation, and usefulness for assessing student learning. A major constraint to the implementation of alternative assessment techniques expressed by teachers is that the techniques often provide data that are difficult to interpret and are

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not useful for assigning grades. However, analytic and holistic scoring rubrics have been shown to alleviate these concerns (e.g., Stenmark, 1993).

This paper reports a subset of findings from an investigation of preservice elementary teachers' beliefs about and knowledge of alternative mathematics assessment. The preservice teachers were involved in a number of mathematics assessment activities and explored a wide range of issues related to mathematics assessment including (a) beliefs about mathematics assessment, (b) the extent to which various alternative assessment techniques can provide important information about students' mathematical learning, and (c) the role of *Assessment Standards* (NCTM, 1995) in mathematics assessment practice.

Methods

Participants and Data Collection

The participants in the study include 61 perservice elementary teachers enrolled in three different sections of a semester-long elementary mathematics methods course at a midwestern university. These students were college seniors engaged in their final semester of coursework prior to student teaching. All of the students completed a series of individual surveys and paired mathematics assessment tasks. However, eight of these students volunteered to be videotaped while engaged in the paired mathematics assessment activities and to partake in an individual audiotaped interview at the end of the semester. All eight volunteers were women between the ages of 21 and 24.

Data were collected from the larger set of participants via four intermittent questionnaires, three assessment activities, and an alternative assessment project. The first survey, a multiple choice Mathematics Beliefs Survey (MBS), was given on the first day of the semester. Subsequent surveys, including an open-ended Mathematics Assessment Beliefs Survey (MABS), an open-ended Alternative Mathematics Assessment Techniques Survey (AMATS), and short-answer/open-ended Mathematics Assessment Standards Survey (MASS), were implemented throughout the semester prior to the coverage of the topic of "mathematics assessment" in the methods course. The three paired assessment tasks engaged the 61 students in (a) distinguishing between "closed" versus "open" mathematical tasks, (b) identifying children's mathematical computation errors, and (c) scoring children's mathematical problem solving via analytic and holistic scoring rubrics. While the rest of the participants were engaged in these activities during class time, the four volunteer pairs were excused and videotaped separate from the rest of class. The researcher elicited a brief videotaped conversation with the four volunteer pairs about the activities upon the completion of each of the tasks.

An additional source of data on the development of students' beliefs and knowledge about mathematics assessment was a class assignment in which all students designed two mathematics assessment instruments, constructed concept maps of "mathematics assessment," and discussed issues related to mathematics assessment. A final source of data were the final individual audio-taped interviews (approximately one hour) with each of the eight volunteer participants. Data were initially viewed by total participant response to surveys and assessment tasks. Later, the eight volunteers' individual cases were examined both on a case by case basis, incorporating data gathered from the paired interactions. In the following sections I report some of the findings from the initial analysis of whole-class survey data, highlighting specific insights from some of the preservice teachers.

Initial Findings

Beliefs About Mathematics Assessment

On the MBS instrument, the preservice teachers were asked 77 questions related to their mathematics belief system including beliefs about the nature of mathematics and mathematics pedagogy. Twenty-one of those questions related specifically to their beliefs about mathematics assessment. Table 1 illustrates the students' responses some of those questions.

Table 1
Percent of students who responded agree or strongly agree to the given assessment statement.

Assessment Statement	Percent Agreed or Strongly Agreed
Written tests are the best means of assessing student progress in mathematics.	18.3
In assessing mathematics, credit should be given to student who solve problems using appropriate method but which differ from the methods provided during instruction.	54.0
Mathematics assessment should include determining students' ability to reason and analyze.	95.1
Mathematics knowledge should be assessed individually.	80.4
Students can assess their own mathematical knowledge.	73.8
Periodic chapter tests are the best means of assessing mathematical learning.	16.4
Mathematics will be easier to assess than language arts.	41.0
Assessing mathematical understanding is easy.	16.4
Assessment of students' mathematical knowledge should yield information about their disposition (attitudes, preferences, etc.) toward mathematics.	34.3
It is important to assign students a number of problems to work in order to practice new mathematical concepts they have been taught.	73.3
Group tests in mathematics can tell you a lot about students' mathematical understanding.	23.0

The percentages of responses by preservice teachers are not too surprising, particularly the beliefs that students' mathematical knowledge should be assessed individually and that it is important to assign a number of problems to practice mathematical skills taught. Also it is encouraging that many of the preservice teachers indicated that assessment should determine students' ability to reason and analyze and that students' should be able to assess their own mathematical knowledge.

In elaborating on these survey responses via the MABS instrument, preservice teachers suggested a wide range of responses to what they believed was the best way to assess mathematics, reporting:

- *I feel pre and post tests can be very effective*
- *through observing their problem solving techniques*
- *writing; if they can write out an answer and show that they understand, then it shows learning*
- *this could be different for each one of my students*
- *in-class work that the students do*
- *by asking them to put down how they came up with their answers*
- *through portfolios so you see how they have improved from the beginning of the year on*
- *I believe that you should use multiple ways*

Only 16.4 % reported believing that mathematics assessment would be easy and 41% thought mathematics assessment would be easier than language arts assessment. When asked if mathematics assessment was the same or different from assessment in other content areas, preservice teachers claimed:

- *math is generally not as subjectively graded. Often times there may be many ways to get an answer but there is only one correct answer. Language Arts is not so clearly right or wrong*
- *I think it is the same in the respect that you are assessing ability and prior knowledge It is different because the content is just different. For example, math is a good subject to do hands on assessing. Other subjects hands-on doesn't seem as practical*
- *math assessment is different than in reading, spelling or social studies. In math students are solving more problems and using more contemplative thought processes*
- *math is more right and wrong with no in-betweens; answers are not opinions*
- *in math you have to understand a lot before you can go on to other problems; you have to know the basics (addition, subtraction, multiplication, and division)*
- *I believe math assessment is becoming similar to other content areas because of the push for writing in math;*

Viewed in conjunction with other expressed beliefs about the nature of mathematics and mathematics pedagogy, these preservice elementary teachers hold a typical range of beliefs about mathematics and mathematics pedagogy (see Raymond, 1996; Thompson, 1992). Because expressed beliefs are not always consistent with teaching practices, and are often challenged in the face of practice (Raymond, in press), it is vital in understanding the development of assessment beliefs to gain a sense of the preservice teachers' knowledge and interpretations of alternative mathematics assessment practices.

A Critique of Various Alternative Mathematics Assessment Techniques

On the AMATS instrument, pairs of students discussed and reported what they believed one could learn about children's mathematical understanding from a variety of mathematics assessment techniques including (a) open-ended problem-solving tasks, (b) quizzes, (c) portfolios, (d) journal writing, (e) teacher observations, and (f) standardized testing. From *problem-solving tasks* the preservice teachers determined that one could learn many things about students including information about how students think, levels of understanding, students' organizational skills, group dynamics, mathematical reasoning and logic, critical thinking skills, different methods children use to solve problems, thinking patterns, students' ability to solve problems, students' knowledge of problem solving and problem-solving strategies, how they apply math knowledge to real world situations, communication skills and creativity. Regarding what the preservice teachers thought they could learn from mathematical *quizzes*, the list is much different. They suggest quizzes demonstrate student recall, understanding of concepts, long and short-term memory, knowledge in particular content areas or section, how well a teacher is teaching, student learning or review of basic concepts, facts, and skills, and how students follow instructions.

As to the benefits and purposes of student *journal writing* in mathematics, the preservice teachers suggested that through student writing they could learn about students' ability to reflect; how students think; individual understanding; student opinions; and writing skills and how well they verbalize. They also believed that journal writing allows students to take responsibility for their own work; to record ideas, feelings, successes and failures; to express both attitude and comprehension; to communicate with the teacher; and to review prior writing about their thinking.

In brief, the preservice teachers demonstrated that they could identify differences between alternative mathematics assessment techniques and the kinds of information about students' learning that one can glean from them. They claimed that having been asked to discern what could be learned from different assessment techniques served to make clear the need for alternative assessment practices.

The Role of Mathematics Assessment Standards

On the MASS instrument, preservice teachers were asked to rank order the six Assessment Standards (NCTM, 1995) from most important (1) to least important (6) according to what they believed was most important to consider when developing a good mathematics assessment plan. In addition to their written explanations of why they viewed one criterion as more important than another, the preservice teachers were later asked to discuss the extent to which different alternative mathematics assessment techniques have the potential to meet the six standards. Table 2 illustrates the preservice teachers' rankings of the NCTM's (1995) *Assessment Standards*.

Of primary concern to these preservice teachers were equity, consistency, and validity and of least concern was openness. Enhancement of learning and emphasis of important mathematics were rated as only moderately important.

Table 2.
Percentages of preservice teachers' rankings of the NCTM Assessment Standards from most important (1) to least important (6).

	Assessment Standard					
Ranking	Important Mathematics	Enhance Learning	Promote Equity	Openness	Validity	Consistency
1	15%	11%	40%	0%	25%	9%
2	9%	21%	15%	4%	13%	38%
3	19%	6%	11%	6%	45%	13%
4	8%	26%	6%	21%	15%	25%
5	28%	19%	26%	9%	2%	15%
6	23%	15%	2%	60%	0%	0%

Regarding equity, most students who selected this as the most important standard wrote that they believed all students should be given a fair chance to show they know the material. On the value of valid and consistent assessment practices, several students suggested, "if the assessment doesn't measure what you want it to, then it is worthless." Students who ranked openness as least important either reported that they were not sure what was meant by "all aspects of the mathematics assessment process should be open to review and scrutiny" (NCTM, 1995) or explained that willingness to allow your assessment practices to be scrutinized was merely a guideline or choice to be made by the teacher, not an integral part of the assessment process.

Some preservice teachers explained their modest ranking of emphasizing important mathematics saying, "all mathematics taught should be important." One preservice teacher deduced, "you need to assess the mathematics that is *taught*, not the most important to learn." Although some preservice teachers expressed that enhancing learning was vital, others made statements such as, "assessment is to gather data about the students' learning and how they learn, NOT to enhance their learning."

Closing Remarks

The investigation explores both preservice teachers' knowledge base related to alternative mathematics assessment as well as their beliefs about mathematics assessment and its relationship to mathematics teaching practice. This research captures preservice teachers' thinking as they develop from students of the mathematics teaching-learning-assessment process to student teachers who must confront mathematics teaching, learning, and assessment issues on a daily basis.

Initial results of this study signify the importance of providing preservice teachers with a forum for discussing and engaging in alternative mathematics assessment practices. Left to interpretation, many of the overarching goals of mathematics assessment reform may be lost to preservice teachers. The examination of beliefs and current knowledge of alternative mathematics assessment practices stimulated these preservice teachers to reflect upon the links between mathematics assessment and instruction.

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BELIEFS OF STUDENTS INTENDING TO BE TEACHERS: ELEMENTARY VERSUS SECONDARY

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This paper examines beliefs of students intending to be elementary teachers (EL group), as compared to those intending to be secondary mathematics teachers (SE group), prior to their teacher preparation programs. The beliefs and attitudes of both groups are also compared to a recent national survey of K-12 practicing teachers and to the goals of the reform movement in mathematics education. Two written surveys, including open-ended responses, imply that the EL and SE groups both have a traditional view of mathematics as a body of facts and procedures, and they lack strong or consistent support for reform teaching strategies. Experiences with mathematics, particularly in elementary grades, have made a negative impression on the EL group. Recommendations include increased efforts prior to and including teacher preparation, especially in revising undergraduate mathematics courses.

There is a growing concern that the current reform efforts in mathematics may succumb to criticism and be replaced by a backlash movement as occurred following the "New Math" of the 1960's. Jack Price recently posed the question "Will we bring thousands of people to join us in building a better mathematics education for all students, or are we talking to each other while waiting for the change that never comes?" (Price 1995). He concludes with the advice to persevere, in that change does not happen quickly. A major problem with the reform ideas being adopted quickly is that traditional beliefs on the nature of mathematics and teaching are inconsistent with those of the reform movement (Cooney, 1985). The traditional view is that mathematics is a body of knowledge or set of rules and facts to be transmitted by the teacher to passive students through memorization and drill. The reform movement adopts a constructivist view that the learner must actively engage in building personal knowledge through relevant thought-provoking activities and reasoning, which are facilitated by the teacher. To evaluate the progress of the current reform efforts, an extensive survey was conducted in the 50 United States and in the District of Columbia in 1993. Weiss' (1995) report on this survey indicates that teachers' responses differ according to the grade level taught, i.e., elementary teachers (grades 1-4) are less confident in their ability to teach mathematics and are far less familiar with NCTM reform documents such as the *Curriculum and Evaluation Standards* (1989) than are high school teachers. However, these elementary teachers espouse higher support for 15 out of 18 teaching strategies consistent with the reform goals when compared to high school teachers. For example, the lower grade teachers advocate "emphasis on writing about mathematics" and "students working in cooperative learning groups" more than high school teachers. This inverse relationship between familiarity with reform documents and support for reform teaching strategies is disconcerting as is the apparent lack of consistent teacher beliefs and practices across grades K-12.

It has been widely reported that teachers' beliefs and attitudes influence their classroom practices and that these beliefs and attitudes are extremely resistant to change (Thompson, 1992). Many studies report on the difficulty of persuading teachers in training to adopt beliefs and practices more consistent with the reform goals (Frykholm, 1995). It has been suggested that the beliefs about the nature of mathematics and teaching are at the very core of the teacher's belief system or make up the teacher's philosophy (Ernest, 1991). Green (1971) describes some beliefs as being central or independent of others and that these primary beliefs are held so strongly and independently that new evidence has little effect on their preeminence.

While a number of studies have investigated beliefs of teachers and preservice teachers at the elementary and secondary grade levels, there is little information directly comparing the beliefs and opinions of these two groups about mathematics and teaching prior to their teacher preparation experience. The concern of this work is to describe and compare beliefs of students who intend to be elementary and secondary teachers that they bring from prior experiences with respect to reform goals. Explicit information about similarities and differences in the core beliefs of these two groups may help to explain the inverse relationship of the Weiss study and help educators plan undergraduate, teacher preparation, and inservice programs to encourage consistent beliefs and teaching practices across the grade levels which are supportive of reform goals.

Methodology and Sample

Several sources of data were used to form a description of beliefs and attitudes of college students intending to be elementary teachers (EL group) as compared to students intending to be secondary mathematics teachers (SE group). It is hypothesized that several indicators will converge on consistent and useful information. The data was collected over a period of two years and includes two written surveys, one with open-ended questions.

Approximately 25 students from each group responded to a 32 item written survey using a five point Likert scale (Ebert & Risacher, 1996). The EL students were generally beginning the first undergraduate mathematics course required for their major (not education majors), and the SE group was just beginning the first semester in the teacher preparation program. This "Beliefs" survey includes statements reflecting beliefs on the nature of mathematics (12 questions), teaching and learning mathematics (12 questions), and the learner (eight questions). It includes both positive and negative statements with respect to the reform goals in each area; for example, "Students should gain practice manipulating expressions and practicing algorithms as a precursor to solving problems," as compared to "problems and applications provide an excellent means to introduce new mathematical content."

The second written survey (Journal) involved approximately 25 SE students and 34 EL students. The survey was conducted at the beginning of the semester in undergraduate mathematics courses required for each group. It should be noted that the teacher preservice program is a graduate program for these students. Students responded to five questions concerning the nature of mathematics, attitudes about mathematics, confidence with mathematics, and influences on attitudes toward mathematics (O'Daffer, P., Charles, R., Cooney, T., Dossey, J. & Schlieack (in press)), with four questions in an open-response format. The students' replies were analyzed for recurring ideas and themes and coded accordingly, for example, "use in solving real world problems" and "very useful in our lives," were both coded as USEFUL with interrator agreement. Likewise, DIFFICULT was coded for "problematic", "complicated" and " hard."

Results and Discussion

Beliefs Multiple Choice Survey Results

Students' replies to the 32 item survey were recorded as 1 = strongly disagree, 2 = disagree, 3 = undecided, 4 = agree and 5 = strongly agree. A number of questions yielded means close to the undecided score of a 3 (means falling between 2.5 and 3.5).

Beliefs "undecided" or Neutral to Reform Ideas: Both EL and SE

- Students learn by remembering what they are taught.
- Students should gain practice manipulating expressions and practicing algorithms as a precursor to solving problems.
- The role of the teacher is to transmit knowledge.
- Mathematical ideas should be examined in terms of a basic formula or standard equation.
- Instruction can best remediate poor computational performance by the deliberate teaching of correct rules.

Beliefs Contrary to Reform Ideas (with means greater that 3.5):

- The best way to model mathematical ideas is to decompose them into a sequence of basic skills which can be mastered one at a time. (EL and SE)
- Doing mathematics involves mastering a set of rules and procedures that may be applied to solve problems. (EL only)

Beliefs of Significant Differences Between EL and SE groups ($p < .05$)

- Instruction should build on students' prior knowledge. (SE more favorable)
- Learning situations should be embedded in authentic problem situations that have meaning for the students. (SE more favorable)
- Problems and applications provide an excellent means to introduce new mathematics content. (SE more favorable)
- Classroom discourse should allow students to experience the opportunity to present their proofs to the rest of the class. (SE more favorable)

In addition, the two groups had similar means in the direction of the reform goals on a number of items; for example, in favor of using realistic situations with messy data, and instruction should promote the development of both conceptual and procedural knowledge.

Open Ended Journal Survey Results

Question	EL Responses	SE Responses
1. Adjectives to describe math	DIFFICULT, STRESSFUL CHALLENGING	USEFUL, INTERESTING FUN, LOGICAL
3a. Like in math?	LOGIC, PB SOLVING	PB SOLVING, CHALLENGING
3b. Dislike in math?	WORD PBS, STRESS	PROOFS, FORGETTING THINGS
5a. Influences +	COLLEGE, HS, USEFUL	SUCCESS, COLLEGE, PARENTS
5b. Influences -	EL grades, HS, WORD PBS	(few remarks, some COLLEGE

When selecting descriptors from a list of "computing, solving problems, applying definitions, reasoning logically , discovering patterns, communicating" the most common replies by both groups were SOLVING PROBLEMS and REASONING LOGICALLY with approximately the same percentages. In response to "How do you feel about your ability to do mathematics?" the groups replied as shown in the chart below:

	Low	Med	Good
EL group	56%	24%	21%
SE group	13%	42%	46%

The results from the two surveys imply that the SE and EL groups tend to have a traditional view about the nature of mathematics as a set of sequential skills and rules with which to solve problems. A common reference by both groups was liking "one answer," "clear rules to follow," and the "orderly" qualities of mathematics. They generally expressed lukewarm or ambivalent support for many of the reform teaching strategies, in agreement with the Weiss study. These beliefs about mathematics and teaching are likely to be strongly held as they have undoubtedly been formed over many years of experience with mathematics and in mathematics classrooms. It is disturbing that these beliefs are so prevalent in persons entering teacher preparation programs today. One note of encouragement is that both groups selected problem solving and reasoning over computing and applying definitions when given only five descriptors.

This work also supports the Weiss study concerning differences in beliefs according to grade level. However, in contrast to the Weiss study, the Beliefs survey indicates the

SE group have more of a reform view of the learner: value students' thinking and interests, prior knowledge, current interests, and the sharing of student explanations, which may encourage instructional decisions consistent with reform goals. However, the significant differences on these items in the EL group may yield a very different kind of instruction during the impressionable elementary school years. It should be noted that the Beliefs survey was originally written with the intent to survey secondary teachers; consequently, it may be that some vocabulary, such as "using formulas or equations" that is more meaningful within the current secondary curriculum compared to elementary. Likewise, the Beliefs survey does not address some pedagogical issues more commonly associated with elementary schools, such as the use of cooperative groups or using manipulative activities, both included in the Weiss survey.

The Journal results clearly give some indication about the very different experiences and attitudes of the EL group as compared to the SE group, but they do not adequately express the tone of emotions expressed by the group: their confidence is very low and anxiety is very high. Unfortunately this lack of confidence in ability to teach mathematics is still present after the undergraduate mathematics courses and the teacher preparation program, implied to the Weiss study. Several members in the EL group elaborated about their experiences, such as "timed multiplication tests" and "long division, for example, "Fourth grade was much more difficult than expected. Math was an utter nightmare because I wasn't really up to speed with multiplication and we were moving on to long division." It would seem that a strong negative influence on the EL group was elementary school experience with mathematics, suggesting an impact on core or deeply held beliefs. EL comments on teachers include, "I has some really awful teachers back in elementary school who thought the only way to learn was to beat me over the head with long division. I started to really hate it. But then in high school and early in college I had some really good teachers and my attitude started to change." Another difference concerns the concept of time: the EL group felt mathematics was "too time consuming," while the SE group said it "takes time and effort." These comments seem to indicate the stress felt by the EL group but not felt by the SE group.

While this study is small and has some uncontrolled factors that limit its generalizability, it highlights some serious concerns of the reform movement in mathematics education. For K-16 students to have a "reform" experience in learning mathematics, the elementary, secondary, and college teachers need to have a similar view of mathematics and beliefs about teaching and learning, consistent with the reform. The results of this study suggest one area in particular need of concern by the mathematics community: the elementary teacher. If we hope for the reform to succeed, we need to consider the negative experiences in mathematics and their impact on the students who will be teaching the elementary students of tomorrow. Increased efforts and new strategies are needed with respect to improving elementary teaching. I would suggest several fronts of action: (1) more active involvement with elementary teachers and inservice programs to ensure early positive experiences in mathematics, (2) a revision of the college mathematics courses for this group with the goal to encourage positive beliefs and attitudes, and (3) revision of teacher preparation programs according to the special mathematics needs of this group. A significant challenge for the mathematics education community would be to revise the college mathematics curriculum for this group. This would imply courses with adequate and appropriate content to teach the reform curriculum, but it also implies teaching by modeling reform strategies. It is hoped that these courses would encourage a reform view of mathematics and more positive beliefs and attitudes. Unfortunately these courses are commonly taught in the "college lecture" format, due to insufficient reform oriented faculty. While some universities may already be offering such a revised program, it is apparent that many of us have much work to do to effect these changes.

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CONTEXTUAL PERSPECTIVES OF SECONDARY SCHOOL MATHEMATICS

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The main aim of this study was to examine factors in students' and teachers' constructions of their world views of the nature and relevance of mathematics. Specifically, the study focused on the following three questions:

1. What do students and teachers think mathematics is?
2. What do students and teachers think are the intentions of mathematics study and why mathematics is included in school programs?
3. What do teachers think students think about studying mathematics s and how do these views compare with those of students?

The study used in-depth, semi-structured interviews with students and teachers at two high schools in the metropolitan region of a large Western Australia city. A total of 40 students, 19 teachers, 2 career counselors, and 2 administrators were interviewed. An interpretive approach was used for data analysis.

Results and Conclusions

Students' and teachers' views about mathematics, as interpreted by the researchers, were composed of four major interwoven and overlapping factors: social status of mathematics, career aspirations, and interest or disinterest in mathematics. The factors were interwoven in that each appeared to be an element in defining each other. Altogether the factors create a web of beliefs that are individuals' conceptions of mathematics and what it means to 'understand' mathematics.

Social status of mathematics: Students' conceptions of mathematics reflected a social norm that mathematics is an 'important' and essential subject to study, with much prestige in the eyes of the community, especially employers. They believed society at large highly values success in mathematics as something for which people should strive.

Utility of mathematics; Students saw mathematics learning as useful because they saw it as something needed in their daily lives, or possibly in future professional endeavors. However, what they often actually described as relevant mathematical knowledge was mathematics taught primarily in elementary school.

Career aspirations: Students were motivated to study mathematics to enhance their prospects for a particular profession or job, to keep their career options open and maximized, or to assist their chances of gaining acceptance at a post-secondary institution.

Mathematical interest or disinterest: A few students expressed an interest in mathematics related to it being a discipline of thinking and they referred to a sense of the intellectual fascination and challenge it provided A strong interest in mathematics was sometimes expressed in relation to career aspirations and the social importance of mathematics, with enjoyment achieved through being successful in relation to these other key components.

The results indicated students put mathematics in a whole school, career and life perspective, rather than a merely mathematics perspective, but mathematics education research does not presently address this broad viewpoint. What is also noteworthy is how the four components identified conflict with the ideals of how mathematics education researchers identify problem s present in mathematics education practices. Recognition is needed for how students view their mathematics experiences because this study indicates students do not separate mathematics from their personal social and psychological contexts. They do not perceive of mathematics as one might describe mathematics as an academic discipline, but rather, they describe mathematics in relation to a range of socially and psychologically derived components.

MAKING SENSE OF BEING A TEACHER: EFFECTS OF BELIEFS ON THREE PRESERVICE TEACHERS

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This study continues a line of research on the development of preservice and first-year teachers' growth in the profession and how that growth is affected and characterized in terms of belief system structure and position relative to epistemological authority. Learning theory particularly as shaped by constructivist perspectives on epistemology, point us to the idea that students learn through interacting with their environment and developing viable constructs for organizing their experience. With this perspective of learning as an adaptive activity, it is important to consider teaching and learning to teach as adaptive as well. The way a teacher adapts is strongly affected by their beliefs and belief structures developed prior to any teacher education experience in which they might participate. In teacher education, there is a need to allow teachers to challenge and examine deeply held beliefs.

We were particularly concerned with the effect of belief structure differences on three first-year teachers' process of learning to teach. The participants were selected from a cohort of twelve preservice teachers enrolled in a one-year teacher education program which follows their undergraduate preparation in mathematics. The participants responded to an open-ended questionnaire, were observed in interactions around the activities of the class. Written artifacts, including journal entries and reaction papers required for the course were collected and the participants were interviewed near the end of the course and again after their student teaching experience.

The three participants exhibited a range of positions with respect to external authority. Edie expected her teacher education to make her a good teacher and accepted various teaching methods and ideas as delivered from external authorities and tended to see teaching as context independent. Brad looked at the world with a guarded eye and insisted that teachers act as arbiters or referees for knowledge claims among students. Karen highly valued her own voice and insisted others listen to their own voice as well. She had originally felt that a teaching career was closed to her but later rejected that notion and returned to teacher education. The participants were all challenged by their teacher education and field experience. Edie was challenged to find her own voice in making teaching decisions. Brad expressed the importance of listening to new ideas and surprised at the animosity he felt from teachers he taught with toward reform. For Karen, the views expressed in her teacher education program excited her as they allowed her to see mathematics the way she saw the rest of the world—tentative, arbitrated by one's self.

The results of this study strengthen the notion that teachers' processes of learning to teach are strongly affected by their existing belief structures and provides insight into the nature of these affects in the context of teacher education experiences.

ONE TEACHER'S BELIEFS AND ACTIONS THAT INFLUENCED THE NEGOTIATION OF A RICH MATHEMATICS CLASSROOM

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The purpose of this paper is to present an interpretation of one teacher's beliefs and classroom actions as he attempted to negotiate a learning environment in a university mathematics problem solving course that fostered "the development of each student's mathematical power (NCTM 1991, p. 57)". Observations of this problem solving course found the classroom interactions to be significantly different from a lecture oriented mathematics course in which the teacher is the dominant player.

Each class session of the semester long course was video recorded to accompany field notes. In addition, the instructor was interviewed in video recorded sessions after each class session concerning his actions and intentions, as well as students' actions and interactions. The methodological stance was similar to that described by Voigt (1989) and Wood (1993) in which "detailed descriptions and interpretations of video recorded classes" (Voigt 1989, p. 28) were used to reconstruct patterns of interaction and routines.

The teacher's beliefs about mathematics and mathematics learning determined his actions in this problem solving classroom and therefore influenced classroom norms. The presentation will be a discussion of the teacher's beliefs about learning, teaching, and mathematics classrooms, as well as his intentions and actions that fostered a problem-centered university mathematics classroom for prospective secondary teachers.

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THE RELATIONSHIP BETWEEN TEACHER AND STUDENT BELIEFS ABOUT MATHEMATICS

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In this study we examine the relationship between teacher (n=7) beliefs about mathematics, the learning and teaching of mathematics and their respective students' beliefs about mathematics (n=162). The data were collected by means of two instruments specifically designed to measure belief systems about mathematics. Teacher scores were adjusted so that a higher score reflected beliefs in alignment with the National Council of Teachers of Mathematics (NCTM) Standards. We found that the student subscales of ego orientation (objective is to do better than classmates), competition, and work avoidance negatively correlated with teacher beliefs about teaching and learning mathematics as reflected by the NCTM Standards. A positive correlation was found with teacher beliefs as reflected by the NCTM Standards and task orientation (objective is to achieve understanding). These findings suggest that this group of teachers practiced what they believed and that these practices affected what their students believed. We suggest that using these two assessments in tandem give a clear picture of the mathematical environment within a classroom and can be used in professional development workshops to initiate teacher reflection about classroom practices.

INVESTIGATING THE BELIEFS OF TWO SETS OF TEACHERS OF MIDDLE SCHOOL INTEGRATED MATHEMATICS AND SCIENCE

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Separation of the content strands has become characteristic of the school curriculum, with separation deepening as one progresses up the grades to high school and beyond (House, 1990). However, there currently is widespread support for teaching mathematics and science in an integrated fashion; integration is advocated as a means by which students can develop deeply organize knowledge structures that are richly interconnected. Despite these expectations, there has been only scant research and experimentation investigating that hypothesis (Berlin, 1991). To address this need, a multifaceted qualitative study was conducted into middle schools in which mathematics and science are taught in an integrated fashion, with the goal of producing descriptive and analytical accounts of two cases of middle school integrated mathematics and science teaching and learning.

In light of current research on teaching which indicates that teachers' beliefs about the discipline(s) they teach affects *what* and *how* they teach (Thompson, 1992; Tobin, Tippins, & Gallard, 1994), one component of this study consisted of an investigation of the beliefs of teachers of integrated mathematics and science concerning: (i) the nature of mathematics and science; (ii) teaching and learning mathematics and science; and (iii) the connections between mathematics and science. In addition, teachers' beliefs about the following were examined: (i) preparation for; (ii) facilitators and barriers to; and (iii) advantages to students when teaching mathematics and science in an integrated fashion. This poster will present preliminary analysis of data collected from questionnaires, interviews, and classroom observations that addresses these issues.

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TEACHING "BOOKS" OR TEACHING "PEOPLE"

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In Taiwan there are 76 vocational colleges. These colleges offer training for becoming technicians in engineering, business, languages, nursing, pharmacy, marine science, and agriculture. These colleges consist of two types of programs; five year study and two year study. The difference between these two programs is one recruits high school graduates while the other recruits middle school graduates. Each year these colleges recruit about 38,860 new students, among them around 25,000 students who have to take calculus in order to prepare themselves for further professional training. The teachers in these colleges, based on their academic background and publications, are divided into four categories: full professors, associate professors, instructors, and teaching assistants. Recently, the Department of Education decided that teaching assistants should not teach in classes and prefers them to do administrational work.

In Taiwan, if one wishes to teach in colleges, one needs at least to have a master's degree. When one gets hired by one college with a master's degree, one automatically gets the instructor certification from the Department of Education. If one has a doctoral degree, one automatically gets the associate professor certification. Once they acquire their appropriate certification, the mobility to an upper level requires producing a certain amount of publications or obtaining a higher degree. In addition to that, the colleges also have to have vacancies for certain positions to allow the mobility.

In this poster, the author is going to describe these college teachers' views about teaching. Usually, the teacher's job is considered done upon finishing teaching one chapter or one section in the textbook. However, the teacher's job is not done, unless the students learned. Thus, the debate is about differing views of teaching; more precisely, are the teachers teaching books or teaching people?

We have highly educated teachers in these professional colleges in Taiwan; however, few of them have taken education or psychology of learning and teaching courses. In conclusion, the author raises several questions which need to be answered. Will sufficiently abundant subject matter knowledge automatically make a good teacher? Do these colleges in Taiwan really need doctoral degree teachers to teach middle school graduates? What does the educational system really want from teachers—teaching books or teaching people?

UNDERGRADUATE MATHEMATICS CLASS AS A COMMUNITY OF LEARNERS: IN THE BEGINNING

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Introduction: The Maryland Collaborative for Teacher Preparation (MCTP) aims to infuse constructivist teaching principles into college-level mathematics and science courses for prospective teachers. A first principle of the MCTP constructivist philosophy is that instruction must account for the beliefs and knowledge structures of students and professors. The proposed poster thus begins its microanalysis of a developing mathematics classroom learning community by comparing and contrasting instructor (the first author) and teacher candidates (n=19) beliefs and course expectations at the beginning of the semester.

Methodology: The data corps for this study consists of participants' journals from the first three weeks of the semester and the classroom videotapes from the corresponding period.

Frameworks: The participants' beliefs and expectations were analyzed along two dimensions: (1) participants as adult learners using a modified scheme informed by Perry Scale of Intellectual and Ethical Development for College Students (Perry, 1981), and (2) participants' pedagogical conceptions using the Ammon-Hutcheson model (Ammon & Hutcheson, 1989).

Summary of Findings: Most students began the semester as dualistic thinkers, believing that the (teacher) Authority knew the (mathematical) Truth. Therefore, they believed that the teacher's role was to transmit this Truth while the students' role is to "work hard" and "keep up with the assignments," Levels 1 or 2 in the Ammon-Hutcheson model. On the other hand, the instructor was operating at the Commitment to Relativism stage, thus viewing his role as helping students examine their thinking through problem solving and other exploration, Levels 4 or 5 in the Ammon-Hutcheson model. We will elaborate how these beliefs and expectations influenced learning and communication at the semester's outset.

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TEACHER CONCEPTIONS OF MATHEMATICS

PRESERVICE TEACHERS' IDEAS OF TEACHING THE CONCEPT OF AREA

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Background

The transformation from college student to mathematics teacher is an astonishing and sometimes terrifying experience. Taken from a socio-cultural perspective, the change from the "college student culture" to the "mathematics teacher culture" presents many challenges to the prospective teacher. Most college students remember traditional high school mathematics instruction, rather than today's classrooms that reflect current educational reforms. A number of researchers have turned their attention to preservice teachers as subjects of study (Brown & Borko, 1992). Some examine preservice teachers' beliefs about teaching mathematics (Cooney & Wilson, 1995; Borko, et al., 1993). Others extend their investigations to include preservice teachers' knowledge (Ball, 1990). In several instances, this work is framed around the knowledge base of teaching developed by Shulman (1986). Eisenhart, et al. (1993) considers preservice teachers' procedural and conceptual knowledge (Hiebert, 1986). Maher and Davis (1990) investigate the interactions between teachers' and students' knowledge. Regardless of theoretical frameworks or methodologies, studies have determined that the subject matter content knowledge of prospective teachers is often weak, incomplete, or dependent upon algorithms and procedural knowledge (Ball, 1990, Simon, 1993; Eisenhart, et al.,1993). Several of these researchers posit that this lack of knowledge affects the ability to employ effective pedagogical content methods or to understand children's ideas about mathematics, and ultimately their development as professionals.

The theoretical framework of this investigation is that of the socio-cultural perspective of Vygotsky (1978, 1986). The importance of social interactions as a mediator of meaning seems appropriate for the challenging interactions that occur between teacher and students in the classroom. Viewed from another vantage point, we assert that preservice teachers are moving from the well-understood culture of college students to a very different culture of teaching in secondary schools. Before student teaching, preservice teachers envision the interactions they will have in the classroom as teachers. Their movement from one culture to another, and the change in the nature of the social interactions from imagination to reality, have prompted us to question how this transformation occurs.

We consider issues related to college students' emergent content and pedagogical content knowledge before student teaching in order to understand how these ideas are constructed or redefined over the course of one's college experience. As part of a larger international study, it is our intent to follow college students through their years of undergraduate study into their student teaching and first year teaching experiences It is through these future investigations that we hope to capture the dynamic nature of learning how to teach within the contexts of preservice teacher education. Because of the integrated nature in which content and pedagogical content knowledge exist in the teaching process, we have chosen to investigate how these preservice teachers plan to interact with their students.

Method

Participants and Setting

Seven preservice teachers, beginning their third year in a four-year teacher preparation program volunteered to participate in this qualitative study. The subjects represented racial and gender differences, and ranged in age from 20 to 24. Three were in the middle grade mathematics program and the other four were preparing to teach high school mathematics. All had taken one introductory mathematics methods course which included 20 hours in a school internship, and all anticipated taking two mathematics methods courses and their student teaching in their senior year.

The setting was a laboratory equipped with multiple resources for planning a lesson. Included were a variety of manipulatives, student textbooks, a mathematics methods text, and a computer with *Geometer's Sketchpad* and a generic drawing tool. Individually the preservice teachers were given the charge of planning a lesson to introduce the concept of area to a mixed ability group of middle school students, assuming that area had not been taught in previous grades. They were given one hour to plan their lessons, and then interviewed about their plans for another 45 minutes. Data consisted of field notes, video/audio tapes, tool artifacts, and the preservice teachers' notes produced in the laboratory setting.

Procedures and Analyses

Initially we identified two strands of the knowledge base for teaching: content knowledge and pedagogical content knowledge. Content knowledge was dichotomized further into procedural and conceptual knowledge. Additional reviews of the data provided more insights into the components of each strand, and these were used for coding lesson plans.

Results

The ideas that college students bring to methods courses and field experiences are rich sources of information. Their past experiences contribute to their understanding of the teaching/learning process, and provide a platform for additional knowledge. In this analysis, we examine the lesson plans of three students in relation to their content knowledge and pedagogical content knowledge of area.

Chris's Plan

Ideas about content and pedagogy. Chris, a 23 year-old white male, explains that he does not know whether he will teach mathematics or go back to working with adolescents in a Church Outreach Ministry. Chris took the previous winter semester off from school to teach snow boarding in Colorado and needs to raise his GPA in order to do his student teaching next Fall.

> *I would begin asking the students what the word 'area' means to them. I think that this is a word that is used everyday and that they might understand like - an acre of land - as a measure of area.*

Continuing to describe his lesson, Chris plans to show a piece of paper, peach on one side and gray on the other. He explains that he will demonstrate that the area of this paper is 'peach.'

> *I will make the point that area has nothing to do with the thickness of the paper. As another way to show this I wanted to use water to cover the surface of the paper, but I couldn't get the water to stay on the paper,* he explains.

His next demonstration related to area involves the roles of length and width in finding the area of a rectangle. *I would still avoid using numbers, but I am working toward having the students' derive the formula for area.* Holding up two sheets of paper, he explains.

> *I would ask which of these sheets of paper has the largest area. Which is the biggest area? Which has the smallest area? The papers make it obvious because clearly the white paper has a larger area than the green paper. Well, next I would hope that someone would put it together that the paper with the largest area also is longer and wider than the green paper. You see, I'm working toward the idea that the length and width contribute to what makes up the area of an object.*

Chris continues describing his lesson by giving his rationale for not giving the students the formula for area. In this description, he stresses that he is a strong believer in the power of discovery learning. Between his demonstration for length and width and the information that area is measured in square units he is confident that the students will come up with the formula.

> *I feel that with a new topic that if you give the students the information, like the formula, then they can take it home and work the problems and pass the test, but they might not learn anything. It's more important for them to learn what area is composed of. I'm a firm believer in discovery learning ... that if the students don't discover it by themselves it won't have as large an impact on them. At this point in my lesson, the students must have figured out that area was related to the length and the width. Then if I tell them that if the measure of the length and width is in centimeters, then the area units are centimeters squared. They're going to realize that you're multiplying two units together, like centimeter by centimeter to get centimeter squared. Hopefully they would come up with what's important - the length and the width. With some luck they will derive the formula that length times width is the area. Once they derive the formula then I would ask them for the area of a square that measured two units.*

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Conceptual Content and Discovery Pedagogy. Chris described his lesson as a simple lesson and noted that it would be challenging to teach a new concept to 30 students at one time. His approach to developing the content of area was different from any of the other plans because of his systematic development of area ideas related to dimensionality, length and width, and units. Chris appears to view this content conceptually, rather than procedurally. The demonstrations that he developed were relatively simple, but powerful. His plan seems to reflect the hope that by covering the components of area in a conceptual way, that students will be able to deduce or "discover" the area formula. For Chris, "discovery" is not necessarily a hands-on experience for individual students. Envisioning demonstrations and questioning as pedagogical methods for interacting with his students, he expects these interactions will provide the context for individual student's discoveries.

Emily's Plan

A double major in mathematics and secondary mathematics education, Emily is a vivacious 20 year-old who maintains a 3.5 GPA. Her scores on the Praxis are very high in all three areas.

Ideas about content and pedagogy. Emily begins her interview by describing that she chose four shapes to teach in her introductory area lesson: square, rectangle, triangle, and parallelogram. Emily will begin her lesson by giving the students a definition of area from the textbook: *Area is the measurement of a region enclosed by a figure. Then I'm going to give them examples, like the area of a desk, and various other shapes in the room like the floor tiles.* Next, Emily will give the students the area formula for a square. *I'll give the formula and then I'm going to give the students an activity. They'll get the formula for each shape before they do the activity.*

She continues to describe what her students will do. *I've chosen at least one activity for each shape using manipulatives for hands-on experience. That's the way I've always learned best - by putting my fingers on it.* For the square and the rectangle, Emily chooses to use the centimeter cubes.

> *I'll ask, What are the dimensions of the largest square you can make? I'd give them time to play around with the cubes and do the same thing for a rectangle. So they will figure out if a rectangle is skinny and long then it may not be as big as one that is more proportionate... by proportionate I mean the lengths and widths are closer in size.* The interviewer points out, *But using 20 centimeter cubes would produce a constant area of 20, right?* At this point, Emily rethinks her position, *Oh, yeah! They all have the same area but the lengths and widths are different. A small width and a long length is a really thin rectangle. That's not going to be the biggest one you know - it's going to be something more like this.* Again Emily shows a 4cm. x 5cm. rectangle. Then she rethinks what she has said again, *It's all going to have the same area but the lengths and widths can vary.*

It appears that Emily is somewhat confused about the concepts of length, width, and the shape of rectangles.

Her description for finding the area of a triangle involves giving the formula [Area = 1/2 base times height]. The students will count and compute using a square cut from centimeter paper. *By cutting along the diagonal of the square, I'm trying to physically show them the reason why area of a triangle is 1/2 base times height.*

Her student activity and the formula for area of a parallelogram is somewhat unexpected. The model which she uses of a parallelogram is composed of a rectangle and two triangles. The formula she plans to give the students is, *The area of a parallelogram is equal to two times 1/2 the triangle base times the height added to the product of the length times width of the rectangle.*

Procedural content and student activities. Emily, while a successful college mathematics student, relies heavily on giving her students procedural knowledge first. She thinks that it is important for her students to "discover" ideas about area, and in the process of planning the pedagogy of the lesson, reveals her weak conceptual knowledge. Discovery learning in this lesson was translated to mean manipulation of materials by the students after being given the algorithm. Emily thinks that these activities will provide students with insights and understanding of the formulas and their meaning.

Kevin's Lesson.

Kevin is a transfer student from engineering who wants to teach mathematics and coach in high school. His grade point average is C+, reflecting some disappointing grades in engineering classes. His writing and spelling abilities are seen as weaknesses by his instructors. At the beginning of his lesson, Kevin plans to give the textbook definition of area to the students. He recognizes that the definition will have little meaning for seventh grade students so he plans to give the definition and have students work in groups to make sense of the definition. Then Kevin describes a student activity for cooperative learning in which students build a rectangular prism of their choice.

> *First, I will pass out the centimeter paper and the centimeter cubes to each group,* he begins. *I'll ask them three questions. The first question is how many blocks are in the box. Then I'll ask for the dimensions of the box, and then how many blocks are there on each side of the box. I want them to make comparisons between the length and the width.* He continues to describe a complex system of reforming cooperative groups to share results of counting the centimeters on the sides of the box. *Hopefully, between those four new people from other groups getting together, one of them will be able to come up with length times width. Then after that I was going to give them a 10 minute lecture where I make sure that everybody got length times width.*

As part of the lecture, Kevin plans to demonstrate the idea of units of measure for area of a square. He takes a square of paper and covers it with nine, square post-it notes. *I'd explain that each post-it is one unit of measure. I would ask the students to count the number of units on the square - there are nine. Then I would tell them that three time three equal nine.* Later in the interview, Kevin says of his lecture, *I want to keep active and do things in front of the students.*

Upon further probing by the interviewer, Kevin defends his use of a three dimensional object for students to explore area because the box has six sides. *Well there are dimensions in the box. In this box there is one to six.* [He appears to mean that here are six faces of the box.] Kevin points to one side of the box that is 2 by 6. *Most students will be able to see that 2 times 6 is twelve.* After more probes by the interviewer, Kevin is able to extend his plan to include a data table. *I would have dimensions and area in columns on a sheet of paper that I would give to the students. Yes! That's what I would do. Make out a table with dimension one, dimension two, and number of blocks.*

Procedural knowledge, models, and cooperative learning. The focus of Kevin's lesson is procedural knowledge with a reliance on cooperative group work and a follow up lecture to assure that students understand the formula for finding area of a rectangle. His choice of two three dimensional models, box and centimeter cubes, may point to some conceptual difficulties he has with the content. Kevin plans to keep his students busy building boxes, talking in cooperative groups, and moving to new groups. There is a strong physical component to the way that he envisions the interactions in his classroom.

Summary

Many preservice teachers need to extend their procedural content knowledge to include the conceptual knowledge of the topics they will teach. While they ascribe to ideas of discovery and cooperative learning, their pedagogical content knowledge for these methods is naive and makes them vulnerable in the future. Finally, preservice teachers need more knowledge concerning the selection, uses, and misuses of manipulative and other instructional materials used for concrete learning experiences

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THE INFLUENCE OF TEACHERS' KNOWLEDGE OF BOTH MATHEMATICAL AND PEDAGOGICAL REPRESENTATIONS ON THEIR CLASSROOM INSTRUCTION: A CASE STUDY IN THE CONTEXT OF ALGEBRAIC MULTIPLICATION [1]

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The purpose of this paper is to examine a case of the phenomenon of teachers' knowledge of representations and its relationship to classroom instruction in the context of algebraic multiplication in eighth grade mathematics. Through a content analysis of the lessons of the textbook related to algebraic multiplication, about 41 main mathematical ideas (content curriculum events) were identified. For each of these content curriculum events, the participant of the study was asked to provide, when appropriate, four types of representations: the definition or symbolic representation, the mathematical proof, a pictorial representation, and a story-problem representation. His classroom instruction was videotaped to examine the relationship between his knowledge of representations and his use of equivalent representations when teaching the corresponding content curriculum events. Findings showed that the teacher's knowledge of proofs was very limited and that his knowledge of the other three types of representations was strong. However, the teacher's use of the representations he knew was limited.

Some researchers (e.g., Fennema & Franke, 1992; Wilson, Shulman, and Richert, 1987) consider teachers' knowledge of representations to be a critical component of teachers' knowledge for effective teaching. Yet, teachers' knowledge of representations is one of the underrepresented research areas in teachers' knowledge. For example, Fennema and Franke (1992) ask "Do teachers know the representations of the content they ordinarily teach? Does knowing these representations make any difference in how teachers teach?" (p. 154). The purpose of this study is to provide some partial answers to these questions in the case of algebraic multiplication in eighth grade by reporting the case of Mr. Kantor, the participant of the study. Specifically, we address the following research questions:

(1) What is Mr. Kantor's knowledge of mathematical definitions or symbolic representations about algebraic multiplication?

(2) What is Mr. Kantor's knowledge of mathematical proofs about algebraic multiplication?

(3) What is Mr. Kantor's knowledge of pictorial representations about algebraic multiplication?

(4) What is Mr. Kantor's knowledge of story-problem representations of each of the curriculum events?

(5) Which of those representations are used by Mr. Kantor in his classroom instruction?

Conceptual Framework

While thinking about mathematical ideas requires that we represent them internally in our minds, to communicate those ideas we need to represent them externally (Hiebert & Carpenter, 1992). Representation are defined by Leinhardt, Putnam, Stein, & Baxter

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(1991) as "physical or conceptual objects or systems of objects that embody mathematical entities or ideas" (p. 89). Mathematical ideas can be represented using several forms such as verbal representations, symbolic representations, pictorial representations, story-problem representations, or physical representations (cf. Fennema & Franke, 1992; Hiebert & Carpenter, 1992; Lesh, Post, & Behr, 1987). Contreras (1996) conceives of the curriculum as made up of content curriculum events. A mathematical content curriculum event is each mathematical idea or object (e.g., concepts, formulas, theorems, axioms, algorithms or procedures) identified in a curriculum or a curriculum text such as curriculum guides or textbooks. Contreras (in preparation) categorizes the representations into two types: mathematical representations and pedagogical representations. Mathematical representations include the definitions or symbolic representations and the proof of the content curriculum events. These representations are general and symbolic in nature. Pedagogical representations, on the other hand, include pictorial representations, story-problem representations, and physical representations. Some mathematics education researchers (e.g., Fennema & Franke, 1992; Hierbert & Carpenter, 1992) believe that the use of both mathematical and pedagogical representations helps students construct both procedural and conceptual mathematical knowledge (Hiebert & Lefevre, 1986).

Empirical Background

Researchers are beginning to examine teachers' knowledge of both mathematical and pedagogical representations. Regarding teachers' knowledge of definitions, Even (1993) asked 162 prospective secondary mathematics teachers to give a definition of a function. She found that most teachers did not provided a definition involving the two most important features of a function: univalence and arbitrariness. With respect to mathematical proofs, Martin and Harel (1989) asked 101 preservice elementary teachers to judge the mathematical correctness of inductive and deductive verification of mathematical statements. They found that many students accepted inductive arguments as correct mathematical proofs. Regarding pictorial representations, Sánchez and Llinares (1992) examined 26 prospective elementary teachers' understanding of the connections between symbolic and concrete representations such as chips and drawings and how the teachers use these concrete referents to explain their procedural steps to generate equivalent fractions. As a general picture, 21 prospective teachers relied on memorized procedural steps for generating equivalent fractions and the remaining five failed to find the answers. With respect to story-problem representations, Ball (1990) asked 19 prospective teachers to provide a representation for $1\frac{3}{4} \div \frac{1}{2}$ using a story-problem or any other type of model. None of the 10 elementary majors and only five of nine secondary teachers constructed an appropriate representation. Simon (1993) asked 33 prospective elementary teachers to provide a story problem for which the solution could be represented by $3/4 \div 1/4$. He found that 23 students were not able to provide an appropriate problem. These studies provide us insights into the nature of teachers' knowledge of representations. However, as suggested by Simon (1993) and Fennema and Franke (1992), we need to examine how teachers use their knowledge in social contexts such as classrooms. This study attempts to begin to fill that gap by examining in detail the phenomenon of teachers' knowledge and knowledge use as described in the research questions stated above.

Methodology and Procedures

Participant and setting. At the time of the study, Mr. Kantor was teaching eighth grade algebra in a school district known for high student achievement and he had been teaching mathematics for five years. The school district is located in a upper-middle class suburban area in a large midwestern U.S city. Mr. Kantor holds secondary certification and his goal is to teach for both procedural and conceptual knowledge. The school district is implementing an innovative mathematics curriculum: The University of Chicago School Mathematics Project (UCSMP). The textbook used was *Algebra* (McConnell et. al, 1990).

Data sources. Data were collected from three sources: (a) the textbook, (b) interviews and questionnaires, and (c) videotapes of lessons taught by Mr. Kantor.

Procedures. We carried out a content analysis of the nine lessons of Chapter Four "Multiplication in Algebra" and the first lesson of Chapter Five, "Division in Algebra", of the textbook to identify the main content curriculum events. A main content curriculum event is one for which the textbook provides representations (examples, definitions, etc.) and explanations. Thirty eight content curriculum events were identified this way. Those events were supplemented with the ones for which Mr. Kantor constructed representations and explanations or students asked questions (three content curriculum events). The 10 lessons were videotaped with the purpose of examining whether Mr. Kantor used the four types of representations for each of the 41 content curriculum events[2]. After the videotaping, interviews and questionnaires were conducted to find out Mr. Kantor's knowledge of each of the four types of representations for almost all of the content curriculum events.

Data Analysis and Results

Each type of representation constructed by Mr. Kantor for each content curriculum event was judged as correct, partially correct or incorrect. Mr. Kantor's knowledge of mathematical proofs was also examined to find out whether he could distinguish between mathematical statements accepted as definitions and axioms.

Knowledge of definitions or symbolic representations. This type of knowledge was examined for about 38 of the 41 content curriculum events. We judged that Mr. Kantor knew the definitions or symbolic representations for about 32 (84%) of the content curriculum events. To illustrate, he represent the associative property as "For $a, b, c \in R$, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$." The representations for the other six (16%) content curriculum events were judged as partially correct. For example, he represented the rule of the signs *the quotient of two negative numbers is a positive number* as "If a and b are negative then $\frac{a}{b}$ is positive."

Knowledge of mathematical proofs. Mr. Kantor constructed about 17 correct proofs, three partially correct proofs, and 15 incorrect proofs of 35 content curriculum events. We found also that he did not have a well developed knowledge about what mathematical statements are accepted as axioms or definitions. To illustrate, Mr. Kantor categorized as definition, among others, the multiplicative property of zero and the rule of the signs *the quotient of two negative numbers is a positive number* . The proof constructed for the rule of multiplication of fractions was as follows,

$\frac{a}{b} \cdot \frac{c}{d} = (a \div b) \cdot (c \div d)$ (This is the same as a divided by b times c divided by d.)

$= (a \cdot \frac{1}{b}) \cdot (c \cdot \frac{1}{d})$ (a times the reciprocal [of b], c times the reciprocal [of d].)

$= a \cdot c \cdot \frac{1}{b} \cdot \frac{1}{d}$ (Drop all parenthesis and regroup.)

$= \frac{ac}{bd}$

This proof was judged as incorrect because Mr. Kantor did not prove the critical step $(1/b)(1/d) = 1/(bd)$. When Mr. Kantor attempted to prove it he gave the following heuristic argument "let's see, how do you go from there [$a \cdot c \cdot \frac{1}{b} \cdot \frac{1}{d}$] to there [$\frac{ac}{bd}$]? This [$\frac{1}{b}$] times this [$\frac{1}{d}$] is the same thing as [$\frac{1}{bd}$]. You can look it this way. You cut something into b parts and cut that into d parts. You cut it into b times d parts." Mr. Kantor was able to construct a proof using previous theorems. For example, he was able to construct a proof for the four rules of signs for division using the corresponding rules of signs for multiplication.

Knowledge of pictorial representations. Regarding pictorial representations, Mr. Kantor contracted about 31 correct representations, 3 partially correct representations, and 2 incorrect representations.

Knowledge of story problem representations. We judged all of the 36 story-problem representations constructed by Mr. Kantor as correct. To illustrate, he created the following story problem for which the solution can be represented by 40R = 600. "Forty bucks a radio. How many can you buy for six hundred dollars?"

Knowledge use. Mr. Kantor only constructed definitions or symbolic representations for about six (15%) of 39 possible content curriculum events. To illustrate, he represented the algebraic definition of division as $a \div b = a \cdot \frac{1}{b}$. Regarding mathematical proofs, he did not construct proofs for any of the 35 possible content curriculum events. With respect to pictorial representations, he constructed pictorial representations for about eight (22%) of the 37 possible content curriculum events. For example, he constructed a rectangle with sides of 3/5 and 3/4 to show that the area is 9/25 and therefore (3/4)(3/5) = 9/25. Regarding story-problem representations, Mr. Kantor only constructed about 10 (26%) of the 39 possible content curriculum events. To illustrate, he used the following situation, among others, to illustrate the meaning of conditional probability, "Six out of 24 people are blonde. . . . seven out of 24 people run track. Out of six blonde people, four run track. P(Run track given blonde) = 4/6."

Discussion

Research literature on teachers' knowledge portrays teachers as having poor knowledge of representations. These researchers argue that we cannot expect these teachers to teach for conceptual understanding. We found that Mr. Kantor' knowledge of representations was strong except in the case proofs. This means that he has the knowledge for constructing procedural and conceptual explanations involving the use of multiple representations. However, his use of both mathematical and pedagogical representations was very limited. From a teaching perspective, this suggests that the relationship between teachers' knowledge of representations and knowledge use in teaching is far from being linear. From a learning perspective, Mr. Kantor missed several opportunities for helping students construct additional internal connections between mathematical and pedagogical representations. As stated by Hiebert and Carpenter (1992), "connections between internal representations [learning] can be stimulated by building connections between corresponding external representations" (p. 66). From a teacher education perspective, these findings lead us to rethink the nature of the impact of what prospective teachers learn in teacher education programs on their teaching practices and how to help teachers use their knowledge during classroom instruction. Based on personal experience, we speculate that Mr. Kantor does not have well-articulated beliefs about the potential of using all these types of representations for teaching mathematical ideas, especially story-problem representations. From a research perspective, we need to understand the factors that influence teachers' use of multiple representations and the extent to which the use of these representations influences students' learning (Fennema & Franke, 1992). All of these areas seem fertile fields for further research into the nature of teachers' knowledge and its relationship to teaching and learning.

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[1] The research reported here is a preliminary analysis based on portions of the first author's doctoral dissertation being conducted at The Ohio State University under the direction of Dr. Douglas T. Owens.

[2] Among the content curriculum events were: the area and array models for multiplication, the commutative and associative properties of multiplication, rule for multiplying fractions, the four rules of the signs for multiplication and division, multiplicative properties of 1, -1, and 0, reciprocals, solving $ax = b$ and $ax < b$, conditional probability formula, the multiplication counting principle, the definition of division, division by zero, etc.

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LOGICAL TRICK OR MATHEMATICAL EXPLANATION: RE-NEGOTIATING THE EPISTEMOLOGICAL STUMBLING BLOCKS OF PROSPECTIVE TEACHERS IN THE SECONDARY MATHEMATICS METHODS COURSE

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What counts as a mathematical explanation? Will any logical "trick" that "gets you the answer" qualify as a mathematical explanation for 5- (-3) equals 8? This study explores prospective teachers' conceptions of mathematical explanation in the context of a secondary mathematics methods course on the topic of explaining integer addition and subtraction. It was found that although all teacher candidates could do integer subtraction and produce a logical trick to explain it, none were able (or inclined) to provide a *mathematical* rationale. It is argued that different presuppositional bases about what constitutes mathematical explanation and understanding are the cause of these difficulties.

INTRODUCTION

What counts as a mathematical explanation? Will any logical "trick" that "gets you the answer" qualify as a mathematical explanation for 5- (-3) equals 8? In this paper, I will argue that prospective teachers' procedural explanations belie instrumental views of mathematics, which must be transformed to a relational view, if this new generation of teachers is to launch the teach for understanding reforms of the National Council of Teachers of Mathematics (1989).

For researchers and teacher educators alike the question of how to re-negotiate prospective teachers' epistemological beliefs and conceptions of knowledge remains. Are epistemological analyses of mathematical concepts sufficient (Sierpinska, 1990)? Are sociomathematics norms the answer (Yackel and Cobb, 1996)? Are descriptions of what prospective teachers bring to the process of learning to teach critical (Ball, 1988)? Recent works by Cobb and his colleagues (e.g. Cobb, Yackel and McNeal, 1992) have shown the importance of understanding (mathematical) classrooms as culture, taking into consideration both sociological/interactionist and cognitive/epistemological perspectives. Nevertheless few researchers have investigated college level mathematics and mathematics education courses for prospective teachers (Ball, 1990, 1992; Simon & Blume, 1994). The proposed paper will attempt to add to our growing understanding of these courses, as they relate to prospective teachers' subject-matter understanding and epistemological beliefs about explanation, from an emergent perspective. Specifically, the paper will address the following questions:

1. What conceptions of mathematical explanation do the participants in this study, both preservice secondary mathematics teachers and their instructor, hold?
2. How are these conceptions related to their notions of mathematics understanding/teaching?
3. How does the instructor negotiate changes in prospective teachers' conceptions of explanation and understanding?
4. What are the implications for the mathematical preparation of teachers?

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CONCEPTUAL FRAMEWORK

What counts as an explanation? What is understanding? What is the relationship between understanding, explanation, and teaching? These questions have been a constant concern in mathematics education where multiple frameworks for thinking about understanding and explanation exist: instrumental/relational (Skemp, 1978), procedural/conceptual (Hiebert, 1986), knowledge *of* and *about* mathematics (Ball, 1988). My theoretical perspective for thinking about understanding, explanation, and teaching derives from epistemology and mathematics education research.

Understanding. To operationalize understanding I draw upon two frameworks. The first is a modification of the *levels of disciplinary understanding* framework of Perkins and Simmons (1988). A scheme for thinking about understanding across disciplines, this modified frame identifies four types of subject-matter understanding beyond the content level of algorithms and rote skills usually taught in schools, including the (1) *concept level* which refers to the conceptual structures that define, bound, and guide inquiry in a discipline (Schwab, 1978), (2) the *problem solving level* which refers to general and domain-specific strategies and heuristic schemas for monitoring one's own thinking, (3) the *epistemic level* which refers to the warrants for evidence in a discipline, and (4) the *inquiry level* which refers to the generation of new knowledge. The second is the Skemp (1978) framework for thinking about mathematical understanding which distinguishes *instrumental* understanding (what and how of mathematics) from *relational* understanding (what, how and why). In the methods course, I promote the view that Teaching Mathematics for Understanding (TFU) entails teaching the four Perkins' levels (what, how, why, what if), whereas Telling Math (TM) touches only upon content and problem-solving levels (what and how).

Explanation. What do we know about prospective teachers' explanations? What criteria for explanation ought prospective teachers espouse if they are to teach for understanding? Prior work on explanation has identified several areas of difficulty that prospective teachers seem to have when learning to explain mathematical concepts and procedures in the mathematics methods course. The major problem is prospective teachers' procedural explanations, which lack regard for meaning and appear to be based on memorization rather than understanding (Ball, 1990).

From constructivist analyses we gain further insight into both the nature of *mathematical* explanations (albeit in second-grade classrooms) and the character of negotiations about sociomathematics norms (Yackel and Cobb, 1996). The results of this work suggest criteria for what counts as an acceptable explanation, namely that mathematical reasons should form the basis of mathematical explanations, that explanations should entail activity on experientially real objects, that explanation is a communicative act and object for mathematical reflection.

Finally, Jane Roland Martin (1970), one of the few philosophers of education to comment upon the relationship between understanding and explanation as they relate to teaching, provides a useful analytic perspective on the characteristics of explanations for teaching that take the form: E is an explanation of X for A. In such explanations, E must account for audience understanding, as well as maintain a deductive relation between E and X. In the case of teaching, this suggests that one's explanation must be adjusted and made understandable for students at different ages and grade levels, while still being deducible from the problem context.

METHODS

The data corps for this study consists of written student (n=21) journal reflections and classroom videotapes of whole class and small-group discussions from the semester-long secondary mathematics methods course. To analyze prospective teachers' notions of explanation, I used the "acts of understanding" methodology (Sierpinska, 1990) which entails unpacking prospective teachers' procedural explanations to identify the epistemological obstacles and new understandings that they encounter and acquire in the course of learning to teach a concept. To analyze the re-negotiation process, I highlight the sociomathematics norms, or epistemic-level insights, that emerge as participants interactively re-constitute what counts as a mathematical explanation.

RESULTS

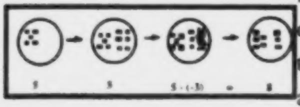
Class discussion episodes reported here are based on this homework assignment.

> Develop an explanation for integer subtraction understandable to eighth graders. Use these four subtraction problems to develop your explanations: **$5 - 3 = 2$; $5 - (-3) = 8$; $-5 - 3 = -8$; $-5 - (-3) = -2$.**
>
> *Prior Knowledge.* You may assume that students understand integer addition using the number line and algebra tile models for the related addition problems: **$5 + 3 = 8$; $5 + (-3) = 2$; $-5 + 3 = -2$; $-5 + (-3) = -8$.** Also assume they know that a yellow tile represents positive one (+1), a red tile represents negative one (-1), and that the *action* of adding more tiles represents addition. Students also understand the idea of zero pairs, i.e. a red tile (-1) plus a yellow tile (+1) equals zero.

Episodes were chosen because they were judged to show significant examples of mathematical explanations and the negotiation process, as well as the epistemological difficulties uncovered by the study. The examples are presented through transcripts of classroom dialogue and summary. We will see that at the outset, course participants disagreed on the criteria for explanation, as well as on the foundational concepts related to integer subtraction. Debate shows they are working from different presuppositional bases about what it means to explain, understand, and teach mathematics.

I. RELATIONAL EXPLANATIONS: SETTING THE STANDARD

This explanation of integer subtraction using algebra tiles sets the standard for explanation in the Teaching for Understanding tradition as it (1) distinguishes symbolic meanings, (2) deduces activity on the tiles from the meaning of the mathematical symbols, and (3) establishes a one-to-one correspondence between the algebra tile microworld and the integer subtraction problem. With Betty's example, the instructor initiates negotiations about what counts as a mathematical explanation.



Start with 5 - (-3). To subtract negative three (-3) from positive five (+5) using the algebra tiles, we must first represent the quantity five using 5 yellow tiles. Now we need to subtract -3 from the five yellow tiles. This means we must take away 3 red tiles from 5 yellow tiles. Since the quantity five as represented (5 yellow tiles) has no red tiles, we need to re-write the quantity five so that it has some red tiles in it. To do this we add three zero pairs, so that we can remove 3 red tiles from the quantity 5 to get the answer is 8.

Instructor: So in this model, the number 5 is represented how?

Class: 5 yellow blocks

Instructor: And subtraction is represented by?

Class: Taking away

Instructor: That's right, it's an action of removing as opposed to addition, which was represented by the action of adding more squares (whether red or yellow). So what's nice about this model I think is that addition and subtraction as operations are distinguished from positive and negative integers by the actions that you perform on the tiles.

By participating in exchanges such as this, teacher candidates learned that the instructor legitimized explanations that DISCRIMINATE the different meanings of mathematical symbols. The concept-level distinctions identified include: operation vs. number, addition vs. subtraction, multiple meanings of the "-" symbol (subtract, negative, opposite) and of the "+" symbol (addition, positive). The concept-level norms being developed here did not come easily to students, as most considered both the + and - symbols to have only one meaning, and virtually all were unaware that context gives the symbol its meaning. At the same time the instructor furthered her pedagogical agenda by legitimizing several epistemic-level sociomathematics norms for TFU explanations, namely: (1) explanations clarify the meaning of mathematical symbols, (2) explanations are deduced from the logic of the problem context, (3) mathematical reasons are the basis for explanations, (4) explanations involving manipulative microworlds discriminate mathematical meanings by symbolizing each mathematical idea differently.

II. INSTRUMENTAL EXPLANATIONS: INSTANTIATING THE STANDARD

Whereas prospective teachers were generally successful in producing meaningful relational explanations for integer subtraction using the algebra tiles, no student produced an appropriate relational explanation on the number line. In this context, prospective teachers' explanations were largely procedural with little attention to previously esta-blished norms concerning the meaning of mathematical symbols, the logic of the problem context, or the relationship between explanations and microworlds. Inability to explain *why* a procedure works was clearly evident. Further, the explanations that prospective teachers did develop to "explain" integer subtraction were often not applied consistently, resulting in many instances of prospective teachers being unaware that their home-made algorithms did not work. Clearly transfer from the algebra tile situation did not occur.

Explanations focused on prior knowledge of the algorithm. Virtually all teacher candidates generated explanations for integer subtraction on the number line that were based on prior knowledge of the algorithm: *subtracting a number from a given quantity is equivalent to adding the opposite of that number to that quantity.* For example, Nikki simply imposed the algorithm on 5 - 3, arguing that we should start teaching where students are. "Since they know integer addition, tell them that subtracting is like adding the opposite. Once they get that down, go back and justify subtraction."

Nikki's firm belief in situated cognition, blinds her to the fact that she has provided *no* rationale for changing the subtraction problem 5 - 3 to the related addition problem, 5 + (-3). A feeling of paradox arises only when the instructor draws an outrageous analogy, suggesting that Nikki's action is no different than changing the subtraction problem 5 - 3 to $5 \div {}^{-}3$, an obviously non-equivalent form. The epistemological difficulty here is that Nikki's action is based neither on the logic of the problem context nor on the meaning of the mathematical symbols. There is no mathematical reason for her action; a correct rule, based on prior knowledge of the algorithm, has simply been imposed.

Directional Explanations. The most common approach to developing an explanation for integer subtraction on the number line was based on the assumption that *symbol - means change direction* and *symbol + means same direction.* The accuracy of this rule depends on the individual's interpretation of it. Feelings of (logical) paradox first arose when Beth discovered a flaw in the rule:

> *Instructor:* What about the linear model? What was your reaction in trying to do that?
>
> *Beth:* The distinction between negative and subtraction is kind of difficult. The problem I ran into was if you did -5 + 3 and you are trying to keep them different directions, I kept seeing that a student would get -8 by doing that because . . . .nothing told them to change directions. I don't know, it just doesn't seem to work as well [as the algebra tiles]. . . [i.e. answer to -5 + 3 would turn out to be -8, instead of +2.]

By participating in exchanges such as this, teacher candidates, who have been sensitized to the characteristics of "good" explanations in the algebra tile context, learn to question the adequacy of this explanation on both logical and pedagogical grounds.

III. TOWARD AN EXPANDED VIEW OF EXPLANATION IN MATHEMATICS TEACHING. Of the directional explanations developed, only one actually "worked" for all integer subtraction problems: *Assume You Will Always Go Forward and Change Direction When You See the "-" Symbol.* Yet the explanation did not distinguish different meanings of the "-" symbol, giving rise once again to the question: What counts as a mathematical explanation? Below the instructor makes the case that Teaching for Understanding requires *explanations to account for why a procedure works, not just how.*

Instructor: [Try the Go Forward rule again for] 3 - (-3); What does that mean on the number line?
Mike: We're considering all of these to be addition problems.
So, alright, ***I can't explain what it means,***
but I can figure out how to solve this using the number line.
Instructor: This is a **great example of teaching at the problem solving and concept levels only.** If you **can't provide a rationale** for the procedure, then until the kids have a rationale,
it's **still teaching a trick.**
Mike: ***Are we using the number line to try to explain what it means,***
or to try and help them solve the problem?
Instructor: Both...
Mike: Really? oh....alright..
I thought that this is just a way to show them ***how*** to do it....
Instructor: Knowing ***how*** to do it, and ***why*** that works are the two things you want to hook together.
TFU requires that you explain *why*, whereas TM requires only *how*.

This episode won the debate for TFU explanations. In this exchange, teacher candidates learned to discriminate between instrumental and relational explanations, founding their explanations on *mathematical* reasons, not tricks. Trained in the Telling Math (instrumental) tradition, teacher candidates had interpreted my request for explanations of integer subtraction to be a request for information about *how* to find the answer. It was a critical moment of profound insight when Mike realized that explanations clarify both how and *why*. With this established, we proceeded in future classes to implement these insights into teacher candidates' instructional planning and teaching.

CONCLUSION

The epistemological obstacles encountered by prospective teachers in making the transition from a TM to a TFU conception of explanation include basic foundational concepts (number, addition/subtraction, positive/negative, opposite) that teacher candidates are assumed to acquire in precollege and college mathematics coursework. One important conclusion to be drawn from this study for the mathematical preparation of secondary teachers is therefore that teacher candidates' prior knowledge of mathematics is not likely to be adequate preparation to teach. This study describes the types of knowledge that prospective teachers need to teach for understanding beyond the content level of information and skills usually taught in schools. Most importantly it highlights the different presuppositional bases that undergird the old and new views of mathematics understanding, explanation and teaching.

This study extends teacher subject matter knowledge research in two areas: conceptions of mathematical explanation and integer subtraction. Furthermore, it offers evidence as to the usefulness of focusing on teacher candidates' procedural explanations as a way of identifying the epistemological limits of their understanding of specific mathematical topics. Finally, the "acts of understanding" methodology pilots a new approach to teacher subject-matter knowledge research, examining teacher candidates, not in the usual interview and written problem-solving research contexts, but in the methods course as they engage with peers and instructor developing and using their knowledge of mathematics to learn to teach.

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DEVELOPING AN UNDERSTANDING OF K – 8 NOVICE TEACHERS: CONTENT KNOWLEDGE, PEDAGOGICAL CONTENT KNOWLEDGE, AND ATTITUDES[*]

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This paper compares the content knowledge, beliefs, and practices of novice K–8 teachers. A comparison is done between a group participating in an NSF grant and those completing the typical program. Comparisons are made from data, pre-intervention baseline to post-student teaching, on their reported teaching practices and on their reported attitudes and beliefs. At the end of their student teaching experience, differences and similarities are found in their understanding of whole numbers, geometry, and fractions; their beliefs about their ability to teach mathematics; and their reported instructional practices. Findings from our data analysis have implications for elementary teacher preparation programs especially the mathematics methods courses and the student teaching experience.

The intent of our work is to foster a deeper and better understanding of the teaching and learning process as related to mathematics and K–8 preservice teachers' pedagogical content knowledge by presenting data from a five-year NSF grant that is designed to better prepare novice teachers. Our data set allows us to provide information on how preservice teachers' pedagogical content knowledge develops during their final year of their teacher education program and through their first year of teaching.

Preservice teachers' content and pedagogical content knowledge in mathematics has been the topic of many research studies. Results of these studies indicate that their content background is frequently poor, and they rarely develop a deep understanding of mathematics (Ball & McDiarmid, 1989; Ball, 1990). During their mathematics methods course and student teaching experience their conceptions of teaching and learning frequently reflect a "show-and-tell" approach. These conceptions too often do not allow them to develop an integrated approach to the teaching and learning experience nor to consider how to assist students to develop a richer and deeper understandings of the content they will teach (Ammon & Hutcheson, 1989). Further, "little is known about the impact of specific experiences on student teachers' pedagogical knowledge" (Jones and Vesilind, 1994, p. 3); i.e. how do their experiences relate to their cognitive reorganization processes? Our work recognizes the growing need (Stoddart, et al, 1993) to develop teacher education programs that reflect recommendations found in current reform documents. Our initial attempts involve providing rich descriptions of preservice teachers conceptions of the teaching and learning process as it relates to mathematics in a K-6 classroom setting.

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During the initial phase of our grant work, university staff assisted experienced K-6 teachers in establishing learning environments that reflect teachers' interpretations of current research findings on how children learn mathematics, particularly in the area of whole number operations, fractions, and geometry. During phase two, preservice teachers were paired with these experienced teachers for both a clinical experience (six hours per week in the fall semester) and a student teaching experience (15 weeks in the spring semester). Two-week summer sessions preceded and followed the school year's work and 20 biweekly meetings were conducted with both preservice and experienced teachers during the school year.

Over a 14 month period, various data were collected on the preservice teachers prior to, during, and following their school year of clinical experience and student teaching. These data included videotapes of model mathematics lessons, stimulated recall interviews of their model lessons, audio tapes of belief interviews, journal entries, and written survey responses. Our data set includes responses from two surveys administered to 21 preservice project teachers before grant involvement and after their student teaching experience, five of whom were specializing in mathematics. The Survey on the Teaching of Mathematics (STM) focuses on the content areas of whole number operations, fractions, and geometry and provides information on teachers' content and pedagogical content knowledge. The Attitudes and Content Survey (ACS) provides information on mathematics courses taken and attitudes and beliefs towards mathematics. During the same period similar data were collected on a group of 17 preservice teachers (comparison group) not involved in the grant work, four of whom were specializing in mathematics. The comparison group had a methods course experience comparable to that of the project preservice teachers. Also, they attended seminars on the teaching and learning process provided by the College of Education. Findings suggest differences and similarities related to content knowledge, pedagogical content knowledge, and attitude towards and confidence in teaching ability as related to mathematics. Our purpose is to present these findings and discuss implications for teacher education programs.

A question from three topic areas of the STM (whole numbers, geometry, and fractions) and ACS questions related to geometry and fractions were analyzed. Some of our findings are reported here.

Whole Number Operations

In the area of whole number operations preservice teachers were given a question involving regrouping for two-digit subtraction involving zero in the minuend. This question asked what they as teachers would do if a student made an error in the computation. Initially the majority of both preservice project teachers (PT) and comparison group teachers (CGT) provided responses that suggested a teacher focus. These included "Show the student what they (sic) did wrong and why what they (sic) did is wrong. Help them (sic) to understand that they (sic) did not regroup in the problem and why it is important that they (sic) do regroup to get a 'right' answer." and "I would explain that we cannot take 8 away from zero, and that we need to 'borrow' 10 from the tens digit in order to subtract."

Initially differences existed between the two groups if the use of manipulatives was suggested in their response. Six of the PT and three of the CGT suggested using manipulatives, however, the PT use was either teacher centered or unclear. For example, "I would use manipulatives to show that their answer is not possible. (Base Ten blocks). This also shows them how and when they should regroup." The CGT use of manipulatives initially was student centered. For example, "The problems students missed would be reworked (in small groups) using manipulatives. Each student would then be asked to draw the problem using figures to represent how they solved the problem using manipulatives." Further initial differences existed in regard to the consideration of students' thinking. Three of the comparison group and none of the project teachers responded similar to the following, "Ask the students how they thought through the problem and try to find out why they got the wrong answer."

After student teaching, changes in the two profiles developed. The CGT continued to be predominately teacher focused in their responses. Further, if manipulatives were discussed in their responses (11 times out of 17) their use was either teacher focused or unclear. Also, no responses considered students' thinking. The PT were split in their responses with slightly more being student focused. Eight PT suggested the use of manipulatives and all uses were either student centered or unclear. Further, eleven PT considered students' thinking when writing down what they would do.

After one year, the use of manipulatives did not predominate the reported instructional strategies of the PT as it did for CGT. Further, the CGT tended to use manipulatives to demonstrate how to do the problems, whereas PT tended to allow students to select the manipulative that best reflected their thinking.

Geometry

One geometry question required student teachers to determine how to help students who were having difficulty distinguishing between area and perimeter. The question provided a given length of fencing and asked how using all of the fencing material in creating different shapes would affect area inside the fencing. On their baseline responses the CGT gave three student-focused responses, the remainder of the responses were teacher-focused or unclear. Twelve of the CGT appeared to have some idea as to the students' problem. The PT had six student-focused, ten teacher-focused, and seven unclear responses. All but three of the responses from the PT indicated some understanding of the problem. After student teaching, the CGT showed little difference among their responses. Many were very directed in their decision. For example, "I would actually give them one solution for one part of the problem."; "Discuss area and perimeter."; "I would draw several models for the class, all of different shapes."; and "I would first have them draw pictures of each different fence. Then bring in the idea of area--discuss area--talk about the different area." The PT responses were all student centered. Typical responses include: "I would ask them to prove it to me. Show me that all perimeters of 64 meters have the same area. I would have them choose two/three different shapes."; and "I would take a long string and tie it together; get some volunteers, and do a demonstration. They would form a circle, a square and a rectangle with the long length and a narrow area." Also, all but one of the CGT limited their pens to rectangles or squares while the PT used a variety of geometric figures including triangles, parallelograms, rectangles, squares, pentagons, and circles. All but three of the PT seemed to understand the problem by indicating in some manner that the students were confusing perimeter and area. Further, differences were found between the CGT and the PT in the areas of perceived grade-level appropriateness of this problem. Initially, the mean grade-level for the two groups was similar, fifth grade. At the end of their student teaching, the mean for the CGT remained the same while the mean for the PT was 4.4.

On the ACS question "What is geometry?", all the CGT responded that geometry is the study of shapes. For example, "Geometry is a lot of things. The first things that come to mind are shapes, measurement, area and perimeter."; "The study of 2-dimensional, 3-dimensional shapes, lines, angles, points, their properties and relationship to each other."; and "Geometry is the study of shapes and their space." The PT responses varied more. The study of relationships, the study of shapes, and other topics were mentioned by most of the PT. For example, "Geometry is working with different shapes and figures, angles, two and three dimensional, points, lines and line segments."; "Geometry is the study of lines, shapes, and all their parts (i.e. line, ray, segment, angle, etc.) It also involves measurement."; and "Geometry is the study of shapes and their attributes (such as angle measure, number of sides and angles, relationship to other shapes, etc.)."

In discussing what K-8 students need to know about geometry, the CGT focused on basic shapes and procedural aspects of geometry, such as, "Perimeter, area, volume, identify and define 2- and 3-D shapes, how to measure items in U.S. units and metric units." The majority of the PT indicated basic shapes, patterns/relationships, some procedural aspects, and relevance in the real world, such as, "I believe in a progression of learning. . . .At first, students need to learn to recognize shapes around them and the diversity of shapes. The students need to break down shapes into their attributes."

Fractions

An analysis of preservice teachers' understanding of fractions from the ACS presented another perspective. After student teaching data responses from both the PT and the CGT indicated a rather limited understanding of content involving fractions. In their responses to the question "What are fractions?", the CGT focused exclusively on a part-whole interpretation of fractions. Only one considered ratios and comparisons. A typical response was short, for example, "Fractions are part of a whole." Likewise the PT also predominately focused on the part-whole description of a fraction; however, many elaborated. For example, several mentioned that "fractions are numbers" or referred to the fact that fractions can be described by the "form of $\frac{a}{b}$ ". None mentioned ratios or comparisons. Although the responses of the CGT were correct they reflect a rather limited understanding of the multitude of ways fractions are used. It is interesting to note that for each group (both PT and CGT), the comments of the mathematics specialists were no different than the non-mathematics specialists in their group.

On the question, "What should K– 8 students know about fractions?", all but three of the CGT mentioned knowing how to do the operations involving fractions. Typical responses included: 1) "K – 8 students should be able to draw, write, add, subtract, order, compare"; 2) "What a numerator and denominator is, how to add and subtract fractions, what a fraction is, that there are manipulatives to help them to visualize what fractions are"; and 3) "What they are, how they're used in life, and how to add them, subtract, divide." The references to "real life" was a common theme among the CGT responses. Several mentioned that it is important for "students to understand fractions"; however, what was meant by "understanding" is unclear in their response. One of the most all encompassing response came from one of the CGT: "Students should be able to draw from his/her informal knowledge to give meaning to a fraction by encouraging students to make connections among fraction symbols, procedures, informal knowledge...helps the children to make a connection." Again, the mathematics specialists' responses of the CGT were similar to that of the CGT in general. It appears that the CGT focus on a part-whole description of fractions only to consider the operations associated with part-whole relationships.

Almost all PT realized that what a K – 8 student should know about fractions snould be more than connections to real life and operations. Over one-third of them did not even mention operations and gave responses such as, "Fractions can represent different things in different problems" and "Students should know how fractions relate to %, decimals, whole number, etc." Four out of five of the PT mathematics specialists focused on the fact that a fraction is also a number.

"What do you still need or want to learn about fractions in order to teach?" was the third fraction question asked of both groups. About half of the CGT used the phrase "how to" in their responses, for example, "How to explain to students why fractions must have a common denominator, when computing." When the CGT did discuss their learning they focused on operations or manipulatives, such as, "I still need to know more practical hand-on experiences to give my students to increase understanding." Only two mentioned the need to learn more about fractions themselves. Over half of the PT explicitly mentioned developing their own understanding of some aspect of fractions, for example, "I guess I still need to work on multiplying and dividing fractions mentally - or without using the equations" or "I would like to further investigate division of fractions and how it works and also multiplication". It is interesting to note that no PT gave a "how to" statement in response to this question. In general, the PT realized their need to develop additional understanding in relation to fractions, whereas CGT were confident in their ability to do fractions and wrote about their need to focus on applications and ways to explain fraction rules during instruction.

Discussion

In all content areas we assessed, PT professed beliefs that students could handle content at least a year sooner that did the CGT. Also, PT were more aware of their own limitations and believed that they could teach mathematics if given the time and if they put forth the effort to develop their understandings. CGT were much more confident in their ability to teach mathematics and less reflective about their own understandings as illustrated by the response, "I believe that my ability to understand math is pretty good because I better understand things when they are proven to me. In math almost everything is proven." A representative PT response was, "I think that my ability to understand mathematics has increased over the past year because I try to look at a problem and how I get a solution not only at the solution." Further, the use of manipulatives played a minor role in developing mathematical understandings among the PT, a major role among CGT.

Implications of these findings relate to pedagogical content knowledge issues found in both methods courses and the student teaching experience. These issues involve, but are not limited to, determining the role of manipulatives in the instructional process, recognizing the ongoing need to learn to teach and learn for conceptual understanding, and developing awareness of young students' mathematical abilities. Our findings provide some insights into these issues.

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POSSIBILITIES AND CONSTRAINTS FOR SECONDARY TEACHERS USING MATHEMATICAL MODELING

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Included in the mathematics education reform effort are calls for teachers to use a mathematical modeling approach in their classrooms. This study involved 21 teachers who attended a four-week, NSF-sponsored, mathematical modeling institute in July of 1994 and then proceeded to incorporate what they experienced into their teaching the following year. There were a number of successes (possibilities) experienced by the teachers that provide hope for the use of a modeling approach by mathematics teachers. Many teachers also experienced struggles (constraints) that sometimes proved impossible to overcome as they attempted to provide students with a modeling experience. This paper outlines what I mean by mathematical modeling in this study and details some of the more significant possibilities and constraints encountered by the teachers.

Calls for reform in mathematics education are numerous, touching nearly every aspect of mathematics content, teaching, and assessment. Mathematical modeling is prominent in these calls. The National Council of Teachers of Mathematics (1989) states that the "mathematics curriculum should include the refinement and extensions of methods of mathematical problem solving so that all students can . . . apply the process of mathematical modeling to real-world problem situations" (p. 137). The Mathematical Association of America (1991) contends that the preparation of teachers should allow them to "(a) work with a given model; (b) recognize constraints inherent in a given model; (c) construct models to analyze real-world settings; and (d) convert among representations (graphical, numerical, symbolic, verbal) that reflect quantitative constraints in a given real-world setting" (p. 6). While modeling is but one of many goals espoused by reformers for secondary schools, a mathematical modeling approach to instruction also meets many other reform goals. Among these goals are students engaged in solving worthwhile problems, using appropriate technology, making mathematical connections, communicating mathematically, solving problems that warrant cooperative effort, and using real tools in the process. However, what possibilities (for success) and constraints (against success) are teachers facing when using a modeling approach?

Mathematical Modeling Institute. The directors of the Mathematical Modeling Project developed it to contribute to the professional development of teachers, and in that way have a positive influence on high school mathematics students. In the summer of 1994, 21 mathematics teachers spent four weeks in the Summer Institute on Mathematical Modeling, a focal point of the project. Most of the teachers arrived with very limited conceptions of mathematical modeling, but were eager to learn. The institute consisted of an introduction to modeling via success stories in modeling; teachers working in groups to solve modeling problems; sessions on technology useful for handling data accompanying real-world situations; workshops on pedagogical issues such as cooperative learning and assessment of project reports; and an extended period for teachers to write projects for use

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in their classrooms. These teachers were from seven states, were nearly split along gender lines, were diverse racially, and had experience in teaching ranging from one to 30+ years.

Description of Mathematical Modeling. I provide a brief description of mathematical modeling as I envision it because I see and hear "modeling" used in very different ways. Mathematical modeling as used here refers to a process of encountering a real world problem or situation, making simplifying assumptions to form an idealized real model, abstracting the real model to form a mathematical model, using mathematical processes to make conclusions from this model, and then comparing the results of this conclusion to the real world problem (see Maki & Thompson, 1973). Typically, it is necessary to perform several iterations of this process before formulating a model that satisfactorily describes or allows predictions concerning the real world situation. In short, I view mathematical modeling as a subset of problem solving—the real-world, "fuzzy" problem subset. Examples of mathematical models include the significant historical models for planetary motion and genetics. Recent work includes use of linear programming for making decisions in business, investigation of the spread of epidemics in medicine, and research into voice recognition in the auditory sciences.

Connection to the Literature. I grounded this study in three literatures: mathematical modeling, teacher change, and professional development. An example of the modeling literature used is the mathematical modeling literature review conducted by Blum and Niss (1991). Included in their work is a list of concerns teachers often have when considering use of a modeling approach (e.g., "Where can I find modeling projects?"). From the teacher change literature, I used the constraints to change (e.g., instructional and environmental) of Duffy and Roehler (1986) and the cognitive requisites to change (e.g., encounter a perturbation) of Shaw and Jakubowski (1991) to assist in data collection and analysis. In the area of professional development, Little (1993) has written that teacher institutes, based, at least, on teacher accounts "offer substantive depth and focus, adequate time to grapple with ideas and materials, the sense of doing real work rather than being 'talked at,' and an opportunity to consult with colleagues and experts" (p. 137). I studied the modeling institute teachers, who had positive perceptions of the institute comparable to the previous quotation, during the 1994-95 school year to look for possibilities and constraints they encountered.

Data Collection. I collected data for this study from a variety of sources. The four-week institute provided an opportunity to observe teachers as students as they worked on modeling projects, experimented with technology, and wrote projects. Teachers completed a detailed questionnaire concerning their background, beliefs, and past practice. During the following school year additional information was collected. Data included a follow-up questionnaire, classroom observations of modeling projects, teacher interviews, artifacts such as student work on projects, teacher reaction to projects, and correspondence such as electronic mail reflections and telephone calls. Many questions for this inquiry came from a pilot study conducted the year previous with participants in the 1993 institute (Preston, 1994).

Discoveries and Tentative Assertions. I categorized findings concerning teachers into *possibilities* (conditions favoring a modeling approach or benefits derived from teachers using modeling) and *constraints* (conditions hindering the use of modeling). Among the favorable conditions were schools open to innovation, schools with technology available for carrying out tedious computations, teachers who can "make time" to do extended modeling projects, teachers who had time to reflect on and improve modeling processes, and students whose enthusiasm and engagement in the projects were high. Benefits derived from the use of modeling included students recognizing the real-life utility of mathematics while they simultaneously began to view mathematics as more than computation, students growing in mathematical communication skills and the ability to function cooperatively, growth over time in teachers' ability to use modeling projects, and involvement of other faculty in the schools.

To give additional detail to this list of possibilities, I describe more fully two of the possibilities—one facilitating condition and one benefit. Several teachers facilitated modeling by overcoming the difficult time constraint, that is, finding time for doing modeling projects. One teacher gained time by skipping the first chapter-and-a-half of the text and simply doing this review throughout the course as needed. It turned out that much of the material did not have to be formally addressed. A second teacher got ahead of others in her school who were teaching the same course by instructing groups of students in different concepts and then having them go back to their regular groups to teach their peers. Other teachers de-emphasized certain topics based on the belief that technological access allows students not to spend as much time on certain symbolic skills as before (e.g., factoring polynomials). A final way that teachers responded to the time crisis was to find projects that tied into the curriculum and to use these projects to replace the traditional treatment of some concept.

Benefits derived from the use of several projects over the course of the year (a facilitating condition) included teacher growth in directing work on modeling projects and student growth in solving the problems they faced. Excerpts from three e-mail messages received from a teacher during the fourth project of the year give indication of this teacher growth:

> *As I do this more, I think that I will more and more come around to the model [of modeling] that was demonstrated for us this summer in the institute. . . . I started teaching modeling this year with the idea to start from scratch and "discover" what things work. I am now convinced of one more thing. If a teacher is going to assign group work, class time must be used to get the groups started. I think that*

> *somewhere down the line after students have some experience, then it may be able to be done completely outside of class. Virtually none of my groups had even started on the projects in my two morning classes today. We used most of the period today, so things are rolling better now. Again, I consider this knowledge for me. You must take them from where they are. . . . I probably mentioned earlier that I now think that the oral presentations are very important. I believe that they should be included as part of the grade. Students should be informed in advance as to what part of the grade will be based on the oral presentation (which I did not do on this one).*

A fourth e-mail message received during the project, indicates student growth:

> *I just had to tell you how great a day it has been. . . . I had SEVERAL students stop in today to ask questions. I was able to "score" quite a few "modeling points". One girl said that she had talked to her dad on the phone about the problem. He told her to fax him a copy of the problem. She said they often discussed tough math problems, but this was the first time he had asked for a copy. THEY came up with the fact that the insurance seemed to be decreasing in value. I told her to explain that in her "Assumptions" and go with it. She showed me how to calculate interest including the current premium and how to add the premium after the interest calculation. . . . A couple of girls came in and had a debate while they were here over the correct way to approach one of the concepts. Either could have been correct. I explained to them that such discussions were a big part of modeling. You are having an excellent discussion and coming to a consensus opinion was a big part of the process.*

I organized constraints into Duffy and Roehler's four categories: curricular, instructional, environmental, and organizational. An examples of a curricular constraint was the presence of a "lockstep" syllabus that some teachers had to follow, particularly those working with other teachers on classes with multiple sections or for those teachers getting students ready for the AP Calculus Exam. Other constraints in this category included modeling projects that were not "teacher friendly" or were too directive. Assessment of oral and written projects turned out to be a difficult instructional constraint as I expected (e.g., see Hodgson, 1995). I coded student resistance as an environmental constraint. The resistance was sporadic and more often than not came from students who had been successful under traditional instruction. Another instructional constraint was the lack of teacher confidence, particularly when facing the decision of how much guidance to give students on projects. Organizational constraints also occurred, such as time: to do the projects, to prepare the materials for the projects, and to grade the project reports.

Some constraints overlapped categories such as teachers and students with limited conceptions of what it means to "do mathematics"—conceptions that caused them to devalue portions of the modeling process. When teachers did this, what was a potentially rich modeling experience became an application exercise, with no real problem solving involved. One such project involved student exploration of whether cars were speeding on the highway in front of the high school. Instead of requiring students to choose the data they needed to collect, the methods of analyzing the data, and the manner in which they would report their findings, the teacher using the task provided step-by-step direction despite suggestions to the contrary from me and several others involved in the summer institute. It was also clear from student work and comments on other projects that they viewed the computational portion of a project as "doing mathematics" but little else. Evidence of this came from student reports that paid little attention to assumptions and analysis of strengths and weakness and from actual comments to the question, "Of the time spent working on the project, what percentage of it would you classify as 'doing mathematics'?" Most of these responses were 50% or under, with comments indicating that processes such as making decisions concerning what mathematical techniques to use were not considered "doing mathematics" by many students.

Discussion. Several conditions facilitating the use of modeling are promising in that they appear to be generalizable. Included in this group are teachers willing to use a number of projects in order get them and their students familiar enough with the modeling process to begin achieving good results and using alternate techniques to allow time for doing modeling projects. Other facilitating conditions are much more situation specific. For example, some schools were well equipped with technology for conducting tedious or difficult calculations, while others had little or no such support. Some schools also provided a supportive environment in terms of teachers willing to work on projects with the teacher who had been to the modeling institute, but other teachers worked in isolation. For those teachers and students able to experience a number of projects (at least three), the rewards in terms of teacher and student growth and appreciation for mathematical power were significant.

Clearly, it is possible to deal with and overcome some aforementioned constraints. In fact, some merely require the willingness to get past the "loss leader" that the first modeling project sometimes turns out to be. However, it is just as clear that there are other, more persistent constraints. Among these are the need for collaborative support from other teachers (all but two of the teachers were from different schools), the realization that because of their busy schedules teachers need to have modeling projects that are in good form, and that teachers need much more assistance in the area of assessment.

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EDUCATIONAL CHANGE IN MATHEMATICS CLASSROOMS: TEACHERS' CONCEPTIONS OF CONTENT, PEDAGOGY, AND EQUITY

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Changes in classroom practices, professional involvement, and knowledge and beliefs of a group of K-12 public school teachers (N=32) who participated in a three year mathematics leadership institute were examined. Data from open-ended questionnaires, interviews, and portfolio boxes substantiated changes in two of the three areas. Changes in classroom practices included increased use of hands-on approaches, less reliance on textbooks, a shift toward teacher as facilitator, and use of more authentic assessment techniques. Increased professional involvement included new and diverse leadership roles, extended professional activities, supporting peers, and conducting training. Changes in conceptions of mathematics and equity were less apparent. There is a need to focus more on the social and psychological dimensions of mathematics educational reform.

The current reform effort in mathematics education has now been underway for about a decade, and studies that report the extent of change in classrooms are beginning to appear (Ferrini-Mundy & Johnson, 1994; McLeod, Stake, Schappelle, Mellissinos, & Gierl, 1996; Webb, Heck, & Tate, 1996). Although documents like the NCTM's *Curriculum and Evaluation Standards for School Mathematics* (NCTM, 1989) and the *Professional Standards for Teaching Mathematics* (NCTM, 1991) have had a significant impact on the field, especially in terms of educational policy (Blank & Pechman, 1995), the evidence of change in classrooms is rather slight (Ferrini-Mundy & Johnson, 1994; McLeod et al., 1996).

The complexities of educational change are now becoming more widely recognized (see, e.g., Fullan, 1993), but we still have much to learn about the process of teacher change. Webb et al. (1996) have outlined a framework for the analysis of teacher change that focuses on teachers' classroom practices, professional involvement, and knowledge and beliefs; in addition to these factors, we have also been influenced by Ferrini-Mundy and Johnson (1994), who focus on changes in the teachers' conceptions of mathematics (see also Thompson, 1992) and conceptions of mathematical pedagogy and by Weissglass (1993) who focuses on equity in mathematics education. We used these frameworks to understand the process of teacher change as it has occurred in a cohort of teachers involved in the San Diego Mathematics Project, a mathematics teacher leadership program.

Method

Following Webb et al. (1996) we analyzed the changes that occurred in 32 teachers from Southern California who participated in the San Diego Mathematics Project Leadership Institute during 1993-96. The group consisted of 27 females and five males with an average of 12 years of teaching experience at the K - 12 level. During the first summer of the leadership institute, teachers attended a three week workshop. Each day began with activities and discussions related to a given topic. During the three weeks the following major topics were presented: what is mathematics, problem solving, children's mathematical learning, school mathematics, constructivist teaching, replacement curriculum, access and equity, assessment, the change process, working with schools, team building, instructional strategies, and networking and empowerment. Part of each day was devoted to grade level and small group sessions that dealt with specific mathematics instructional materials or strategies and selected research topics. Two months later participants attended a two day retreat that focused on curriculum and instructional programs. Two follow-up days were devoted to leadership development, facilitation skills, and equity in mathematics education. The second year of the program included a two week summer workshop and five follow-up days using a format similar to that of the previous year. Major topics addressed during that year included educational change, geometry, equitable instructional practices, parent and staff development, assessment, and algebraic thinking. During the third year the following topics were presented either at the summer workshop or at follow-up sessions: sustaining and nurturing leadership, teaching second language learners, assessment, discrete mathematics, adoption of curriculum materials, and performance standards. Data were collected through questionnaires, interviews, and portfolios that were prepared by the teachers.

The questionnaires were administered at the beginning of the project and annually thereafter, providing a broad overview of changes in the teachers' reports of their professional practice. Questionnaires included reports of teacher beliefs and opinions related to mathematics content and pedagogy, as well as demographic data. Interviews focused on how the project had changed teachers' patterns of instruction and influenced their leadership development. Teachers were invited to maintain a portfolio box with a collection of items that illustrated the kinds of change that they had undertaken. Portfolios included an extensive array of artifacts (including, for example, 5 to 50 items in each of the portfolios), providing evidence of materials that teachers had developed and instructional innovations that teachers had attempted.

Results

The analysis of responses from the administrations of open-ended questionnaires and interviews is presented in relation to the elements of professional growth as outlined by Webb et al. (1996).

Classroom Practices

At the end of the first year teachers reported using more student centered approaches and taking on the role of facilitator rather than the traditional teacher role.

> "I am trying to be more of a facilitator and a resource person, rather than a direct instructor."

> "I try to be the facilitator and let students figure stuff out by debating problems with their group instead of giving them answers or lectures."

> "The project has made very sensitive and concerned about the linear mentality that my generation has experienced in the way we were educated. I've taken my first few solo steps away from the conventional, thanks to the support and resources of the Math Project."

After the second year, participants reported feeling more comfortable using a hands-on approach to teaching and relying less on a textbook. They increased the use of replacement units, investigations, and student writing in their classrooms and continued to evolve as facilitators of instruction.

> "I have moved farther and farther away from the text driven/computation only curriculum. The project has helped to give me confidence to stretch professionally."

> "My classroom mathematics lessons, investigations and projects involve use of writing, literature, and children choosing and using appropriate tools. These constructivist approaches are the basis for lessons that empower students."

During the third year, 81% of the participants reported changes in their assessment practices. They reported using a variety of assessment techniques including student journals, open-ended questions, teacher observations, student projects and portfolios.

> "I try to use authentic assessment activities. I don't base their grade on written tests only. They are also assessed on group work, presentations, and reflective writing activities."

> "I use a wider range of techniques - - involving students in developing rubrics, self evaluation, and peer evaluations."

Professional Involvement

Changes in professional involvement during the three years fell into four categories: new leadership roles, extended professional activities, supporting peers, and conducting training. Leadership activities included being selected as a mentor teacher, serving on textbook adoption and curriculum committees, and being a scorer for the state mathematics assessment program. Professional activities included joining and becoming more active in the local and state mathematics professional organizations, attending more district and county mathematics workshops, and presenting papers. Supporting peers in their development as mathematics teachers included both formal and informal meetings where references, materials, and activities were shared. Conducting training included presenting Family Math for parents and replacement curriculum units for teachers.

> " I have tried to extend teachers' definition of math beyond 'number'. Change is happening, but slowly. I am presently looked at as a 'math expert', but I don't like that definition. I want to be thought of as a resource with resources."

> "Site principal and staff look to me as a knowledgeable math leader. Because of the San Diego Mathematics Project, I have more credibility - also district acceptance of me as a leader."

> "I have become more of a cheerleader and resource person. I have lots of ideas and beliefs that I did not have before. I coach my colleagues rather than strive to change them to my view."

By the end of the third year over half of the teachers had made presentations or conducted workshops at local and state mathematics education conferences, and one third had become mentor or resource teachers and were providing support for their peers on a regular basis.

Knowledge and Beliefs

After the first year participants' writings reflected less anxiety and more conceptual understanding of mathematics.

> "If I were taught with the methods we investigate here at the project, it probably wouldn't have taken me 35 years to get over my math phobia."

> "I am learning more mathematics than I ever did in high school and college. Sometimes it is difficult for me. I can certainly empathize with my students. This is such an exciting time to be teaching mathematics"

> " I'm learning the 'process' , how to facilitate the information. Experiencing permutations this summer and knowing that it is not a mystery made me feel that I do know math."

Studying topics such as fractals, discrete mathematics, and the use of technology was motivating and helped develop the mathematical confidence of the teachers.

> " In the two institutes, I have become so inspired! I realize how weak math was for myself and how that tailored my accomplishments."

> "Fractals and chaos theory -- WOW. I'm hooked! I can't wait to get my kids into this -- making connections."

> "I like the idea of teaching 'cutting edge topics' (fractals, high tech). Students and I are more motivated, and the requisite math can be recovered as needed."

In order to try to measure how the changes that teachers were reporting were reflected in their classroom practices, portfolio boxes were distributed with instructions to provide things that were a part of daily teaching activities. That supported change made in leadership role, instructional strategies, assessment of students, equity awareness, and integration of technology. The classification of the artifacts in portfolios from a subsample of 11 teachers is presented in Table 1. The number of artifacts in a portfolio ranged from 5 to 50. The fact that the category with the most artifacts is leadership role is not surprising given the leadership focus of the San Diego Mathematics Project. The lack of artifacts in the equity awareness category is interesting given the amount of time devoted to the topic of equity in mathematics education during the three years of the project.

Table 1. Summary of Contents of 11 Portfolio Boxes

Category	Number of Artifacts
Leadership Role Samples (i.e., letters from principals, staff development programs, interview notes, and peer coaching notes)	198
Instructional Strategies Samples (i.e., lesson plans, handouts, photographs, student journals, and posters)	136
Assessment of Students Samples (i.e., student portfolios, case studies, student reflection notes, rubrics, and teacher notes on student progress)	59
Equity Awareness Samples (i.e., equity conference materials, research notes, materials for sheltered instruction, and presentations)	22
Integration of Technology Samples (i.e., hypercard stacks, videos, software, e-mail messages, and photographs of students using technology)	26

The data indicate substantial change in teachers' classroom practices, ideas about pedagogy, and professional involvement, but less change in knowledge and beliefs, specifically teachers' conceptions of mathematics and in approaches to equity in the classroom. Most teachers showed evidence of the changes in the major areas outlined by Webb et al. (1996). In line with other research (McLeod et al., 1996), the data indicate that changing teachers' conceptions of mathematics is a long and difficult process. In the area of equity, teachers indicated an awareness of the issues, but there was little evidence that there was any change in classrooms that would have a significant impact on equity in mathematics education.

As Weissglass (1994) has noted, educational reform is a multidimensional effort in which the social and psychological dimensions have been largely neglected. He advocates the use of methods that systematically address feelings, emotions, and values. More substantial support for teachers may be needed if reformers are to have an impact on areas where teacher conceptions are difficult to change. Weissglass (1993) outlines a strategy whereby support groups can be developed in ways that may be more successful in promoting change in equity and other areas where reform in mathematics education is proceeding slowly, if at all.

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TEACHER DEVELOPMENT

CONNECTING REFLECTIVE ACTIVITY & ORIENTATIONS TOWARD AUTHORITY: PRESERVICE SECONDARY MATHEMATICS TEACHERS' VIEWS OF MATHEMATICS

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Expanding theories of learning support mathematics educators' design of teacher education programs that promote professional growth through reflective activities focused on mathematics, learning, teaching, and knowing one's self. Framed with the notion that habits of mind influence development, this study investigates preservice teachers' patterns of reflection and orientations toward authority in mathematics as related to their professional development. Preservice teachers oriented toward external authorities in mathematics tend to accept mathematics as static truths and focus on remembering rather than seeking connections and deeper meanings through reflective thought. Teachers oriented toward a combination of both external authority and internal voice engage in reflective thought that promotes professional growth.

Introduction

As an enabling agent in intelligent action (Dewey, 1933), an activity crucial to the artistry of teaching (Schön, 1987), and a time "when the interlocking wires of constraint and possibility are loosened and we act, even if tentatively, to meld struggle with potential" (Brunner, 1994, p. 47), reflective thought is highly promoted in teacher education. A better understanding of patterns of reflective thought provides insights that may ultimately help instructors facilitate preservice teachers' professional growth. An orientation toward multiplism rather than dualism and a respect for uncertainty adjoined with a sense of responsibility suggest a willingness to question interpretations and engage in self-reflection conducive to becoming a teacher (Brunner, 1994). This study is a search for a deeper understanding of the relationships between orientations toward authority in mathematics and patterns of reflection.

Questions, Methodology, and Theoretical Perspectives

The research questions are situated within preservice secondary mathematics teacher education and limited to implicit and explicit views of the nature of mathematics.

1. What are preservice teachers' orientations toward authority with respect to mathematics?
2. What are preservice teachers' reoccurring patterns of thought accompanying reflections?
3. How are each preservice teachers' orientations and patterns of thought related to the emergence of reflective activity conducive to becoming a teacher?

The four undergraduate preservice teachers selected for this study were part of a more extensive research project during their teacher education program. Participants for this study were selected from two cohort groups. Selection was based upon preliminary analysis suggesting diversity of orientations toward authority in mathematics. Orientations included tendencies toward dualistic and multiplistic thought and ascriptions toward authority as relegated to external sources or as a confluence of external and internal voices.

The primary data were collected through an initial survey, five equally spaced interviews during the program, and field observations and follow-up interviews common to both cohorts. Reoccurring patterns of thought unique to each participant emerged from data associated with "reflection." Data with reference to previously constructed knowledge about oneself or re-presentations of an experience comprised the reflection data. This reflective data as well as other data were analyzed to gain greater insights into the preservice teachers' orientations toward authority in mathematics.

Although notions of closed- and open-mindedness, permeability of beliefs, and structures of beliefs informed initial analysis, aspects of Perry's (1970) scheme of dualism, multiplism, relativism, and commitment were central to the in-depth analysis. Perry's scheme moves from reliance on single external authorities, through recognition of multiple authorities, through differential valuing of authorities, and finally to the establishment of personal commitment based upon an acceptance of an internal authority. Belenky, Clinchy, Goldberger, and Tarule (1986) provided additional perspectives on orientations toward authority in knowing through received, subjective, procedural, and constructed knowing. Orientations toward external authority are prevalent in received knowing and orientations toward an inner voice in subjective knowing. Tendencies toward external and internal authorities are held as either separate or connected in procedural knowing and as integrated parts of one's own knowledge in constructed knowing. These theories provided insights into the orientations of Henry, Sally, Carl, and Shannon.

Introducing Henry, Sally, Carl, and Shannon

Henry's expansion of his survey response, "A function is an equation whose graph has no x values that are equal" was "an equation that you could input a value and get a value out of." When hopes of moving him past this limited perspective dimmed, I tried to move him toward reflective thought by asking if there was a reason for knowing about functions. He responded, "To tell you the honest truth, unless I'm being ignorant to what the definition of a function is, I couldn't see why you would need one outside mathematics or an engineering type field." Henry's narrow view of mathematics hinted at his dualistic orientation.

Henry expressed displeasure when asked to explain what mathematics is. He said, "I don't like that question!" and then he added, "Mathematics is just this fast world of how things are done numerically....You have to be able to use math." He talked of mathematics as a set of facts that were given to us from external authorities. A mathematics teacher was like a "news broadcaster who tells the story like it is" and students receive it and "practice it." He did not make the familiar strange by re-presenting his mathematics and viewing it critically. He had no mathematics of his own to re-present and no reason to question what he had. The idea of multiple strategies was foreign to him. After being prodded to consider multiple problem solving strategies, he asked to move on adding, "You squeezed two out of me." He lacked a multiplistic orientation and connections that would allow him the latitude to move between ideas and reflect thoughtfully. He was unable to make the strangeness of reflective thought familiar. Mathematical reflections were out of reach.

Sally elucidated her response to "A function is" by explaining, "I put 'It is a dynamic expression' mainly because I couldn't think of the real definition and I know in my head what it is but I'm not very good in saying things in words sometimes." Sally described the experience of taking the survey and not being able to recall the real definition. Her need for "the real" definition supported dualistic tendencies. Then in her subjective way of knowing, she drew from within and shared a self-concept built upon personal experiences. She followed with a third grade experience reinforcing a view of mathematics as a received set of definitions and models, "When you have a box and you put something in the box and something else comes out and that's how I think of functions still." She included an internal voice as she reasoned but little reflective thought was provoked.

But reflective thought was provoked on other occassions. When reviewing her example of crewing as a first-hand application of function, Sally became puzzled. After a long pause she explained, "Well, this is kind of physics and you won't use it [mathematics] as much but ... there's a lot of different factors that went into how the boat would row and how fast it would go.... We thought about how many strokes we were doing per minute, we'd have to do more or less because the water's going faster or slower .... We wouldn't say like the velocity of the water is." Mathematics inherent to rowing was inconsistent with her view of mathematics as computation. Sally's multiple views of mathematics and the difficulty she experienced as she tried to make sense of her external and internal orientations toward authority, incited perturbation. Sally also became anxious when reflecting thoughtfully on problem solving strategies. She had argued that if an answer existed, one needed to find a definite answer by a method other than trial and error. Yet, when explaining how she solved the subsequent problem, she laughed and said, "So it [trial and error] is okay in this problem but it's not okay in that problem." Sally's separate procedural way of knowing was influenced by her subjective voice and her received knowing. Sally was easily incited to reflective thought but her lack of connections to others' ways of thinking thwarted her efforts toward resolution.

Carl's comments regarding his incomplete survey responses relating to functions began with, "I forgot the math that could tell me... I didn't know exactly how to go to the next step to solve it." Later he mentioned "Anytime I can't come up with an answer to something.... the first thing I do when I have an opportunity is just to find a resource that gives me the answer." His reliance on external authority in mathematics supports his idea of "math as rules." Although not a proponent of pure memorization, he believes that "basic skills are a form of memorization which is a solid state that you have to begin with. It's the foundation. The applications are to help you remember, make a connection between those basic rules and how to use them later on. You may not remember the rule but hopefully your application ... will click in their mind and relate back to the rule." When considering specific mathematical activity, his reflections were very limited. When Carl spoke in general about mathematics and learning, he referred to mathematics as "thinking skills" and "problem solving" and he spoke of connections with a multiplistic orientation. "Connections to me is going at the same problem and using different directions."

Carl's exhibited two different worlds through his reflective actions. When discussing theory and practice, his flamboyant narratives echoed problem solving, reasoning, communication, and multiplistic views of mathematics and teaching. His teaching did not support these views. When reviewing problems he focused on his singular strategy and disregarded alternative strategies suggested by students. As he recalled one student interaction, he explained that he did hear the student but did not understand what she meant, that he "did not connect with her thinking." It was when his cooperating teacher attributed his unsuccessful teaching to his shaky mathematics that he acknowledged his lack of connections within mathematics and began to question his readiness for teaching. He realized that his lecture and quiz methods were not consistent with his goals. "I lost sight of my goals." His worlds of theory and practice collided and he entered an extreme state of disequilibrium. Rather than struggle with his mathematics to alleviate the tension, he jumped at the opportunity to use an external authority and the timely notification of his grade on the state teaching exam, to reaffirm his credibility in mathematics.

Shannon's views of mathematics as investigation and "a good bit of trial and error" and not as memorization strengthened as she learned how to design and facilitate classroom activities that promoted conjecturing and critical thinking. When reflecting on her survey responses, she related her traditional experiences with school mathematics but argued that mathematics was least like an assembly line "because I don't think that's the way teachers really want you to learn, just doing exact problems. They want you to learn something from the original problem that you can carry on and you know, make something new." She acknowledged a need for a firm foundation in mathematics and described mathematics as "exploring and challenging. It's not cut and dry. It's not this step, this step, then this step."

Reflecting on "a successful lesson", she described how she encouraged a "deeper understanding" of addition of fractions by teaming students and asking them to be both teacher and student as they used colored blocks to model the addition of fractions. Her assessment of students' written explanations directed toward helping a blind person understand, indicated that her students had not only gained a deeper understanding of fractions but were able to make a transition from their activities with the manipulatives to an abstract verbal description of the operation. Her teaching and follow-up reflections were indicative of her relativistic orientation toward authority in mathematics. Shannon respected and encouraged students' individualized problem solving strategies and critical assessment of those strategies. She encouraged inquiry by dedicating entire class periods to students' making sense of problem situations involving seemingly unfamiliar mathematics. She promoted the inclusion of an internal voice in mathematics. She directed her students with "You're not going to know how to do everything right off the bat but if you use what you do have then maybe [you can] come up with some reasonable way to look at it."

Shannon's reflections were focused and holistic. She included initial reactions, expectations, and multiple perspectives interspersed with critical assessment and possibilities for betterment. A notion of continual change wove through her reflections, from function as representing a relationship denoting change to her quest to keep attuned to new ideas in mathematics education. She attributed much of the expansion of her own thinking to her reflective practices and "who she is" to life experiences. Awareness and dedication to meeting the emotional needs of her students also permeated her reflections. She acknowledged that she is very adept at putting herself in her students' shoes and is committed to designing student empowering classroom situations. She gathered ideas for activities from a variety of resources including her students.

Patterns of Reflection and Orientations

This research supports the notion that opportunities for reflection do not ensure engagement in reflection. It also supports the notion that engagement in reflection does not ensure professional growth. Although Henry's reflective thought was hampered by his dualistic orientation and revealed no pattern of reflection, the research has revealed a distinguishing feature or pattern of reflection that dominated the reflective thought of each of the other participants. In these cases, reflective thought was accompanied by movement, a movement that interrupted habits of mind by allowing inconsistencies to surface or by awakening voices that may have been silent in the past.

The pattern illumined by Sally was sparked by her attention to inconsistencies as she reflected upon her personal mathematics and her school related external mathematics. Her growing comfort with a multiplistic orientation and subjective knowing sustained her attention. Her awareness of inconsistency was so overwhelming that she decided to postpone her teaching career until she could make sense of things.

The pattern distinguishing Carl's reflective thought was his movement to comfortable domains within his reflections. He was comfortable in his dualistic and external authority oriented world of mathematical practice and comfortable in his multiplistic and internal authority oriented world of theory. He kept these domains clearly separated until, with the help of caring others, he came to see his worlds simultaneously and in conflict with one another. His initial plans to bridge gaps in his mathematics represented a possible move toward an orientation of multiplism and integrated external and internal authorities but these plans were quickly tossed aside in favor of an external authority.

The prominent pattern in Shannon's reflective thought was her continual drafting of a vision, a vision grounded in an integration of experiences and focused on teachers as agents of change. Expansion of her critical thinking accompanied her movement from a multiplistic to a relativistic orientation. Shannon's reflective thought flourished in an environment that awakened her inner voice and supported an orientation that included an internal authority in mathematics. Unfortunately it was not easy to detect this pattern until her voice was heard.

A Final Perspective

Preservice teachers' patterns of reflective thought vary widely within the same educational program. This study has revealed that within the mathematical arena alone, orientations toward authority are related to varied patterns of reflection and are influential in professional growth. Beliefs, knowledge, and other aspects of the persona of teachers have also been studied in light of their professional development. We might use insights from research into what teachers bring with them to their educational programs as we engage in the study of teacher education experiences that influence patterns of reflection and professional growth. We might address issues from a slightly different perspective by asking different questions. How might we as mathematics educators, help students like Sally get beyond the struggle? How might we design programs to help the Carls sustain movement toward reconciliation of different worlds? What is it in the configuration of an activity that awakens an inner voice? And while inner voices are gaining strength, how might we facilitate an integration of voices that promotes professional development?

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EXTENDING THE EDUCATIONAL CONVERSATION: SITE BASED STAFF DEVELOPMENT

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This paper reports on a study involving department chairs and 10 principals from 19 schools involved in a high school staff development project, "Building Bridges to Mathematics for All." In past research related to this project, we have used focus groups, quantitative data, teacher interviews, and principal interviews to ascertain the impact of the project on teacher beliefs and classroom behaviors. In this study, we examined school designed inservice programs and interviewed mathematics department chairs and selected principals to determine their views of the effect the project had had on their teachers' attitudes and pedagogy and on the mathematics curriculum at their schools. Results show that sustained support is felt to be critical to implementing "algebra for all" and moving schools toward reform in their mathematics curriculum.

Related Literature

Although in the 1986 edition of the *Handbook of Research on Teaching*, the chapter on mathematics education (Romberg & Carpenter) hardly mentioned research on teacher education, our knowledge of mathematics teacher development has progressed considerably in the last decade. One of the critical issues being studied relative to staff development for teachers is how one creates a milieu which fosters change in teacher behavior in the classroom. A number of researchers have highlighted the importance of teacher beliefs to change in teacher behavior (Cooney & Jones, 1988; Ernest, 1991). As late as 1988 Grouws pointed out that there was little information available about the overall design features of inservice education programs which produce changes in teacher beliefs and classroom practices. He called for careful studies which focus on the impact of various features of inservice education on classroom practices.

While connections between teachers' beliefs about mathematics and their classroom behavior have been made (Ernest, 1991) beliefs may also be influenced by other factors in the context of the school and the classroom. As Cooney (1993) has pointed out, there are several metaphorical ways of examining teacher beliefs. Considered in different ways, teacher beliefs might seem contradictory with classroom practice. This view of beliefs has informed our work as we have tried to ascertain not only what works to foster teacher change, but also what impedes it in the school setting.

Background

Since 1992 we have been engaged in an extensive staff development program with high school mathematics teachers, "Building Bridges to Mathematics for All." The program, associated with a national program entitled Equity 2000, focussed on new curriculum, updating pedagogy, and examining issues of equity. As part of Equity 2000 involvement, all of the schools in the two participating districts had committed to place all ninth grade students in a course at least as high as algebra 1/course 1 by Fall 1994. In addition to extensive staff development intended to reach all of the teachers, "Building Bridges" provided classroom coaching and purchase of mathematical materials such as graphing calculators and computer software to support instructional innovations in the classroom. The program is discussed in more detail in Peluso, Pence and Becker (1994).

In three small-scale studies (Peluso, Becker, Pence, 1996; Becker, Pence & Pors, 1995; Peluso, Pence & Becker, 1994) we have endeavored to evaluate the impact of the project on participant teachers. In particular, we have been interested in identifying specific aspects of the staff development which have helped to change teachers' beliefs about the teaching and learning of mathematics, and which have stimulated concomitant changes in classroom practices.

From these studies, we realize that project efforts are effecting change but the impact across school mathematics faculties is varied. While many teachers are slowly making changes in their teaching and a smaller number are leading the way through their example as change agents, others are reluctant to change. In an attempt to support the leadership work in the schools, an extension to cross-district inservice focused on schools. Each participating school was provided the opportunity to design and carry out, with project staff support, site-based teacher development. This site based professional development was individually tailored to the needs of the school and its mathematics staff.

In this study we examine the school level professional development. As the schools plan and carry out their individual staff development, we are interested in the issues addressed by the schools, the way in which they choose to address them and the perceived impact. This provides another lens on the aspects of staff development which help change teacher beliefs about the teaching and learning of mathematics and the related changes in the classrooms.

Methodology

This study includes the review of nineteen school based staff development projects. A kick-off meeting for all of the interested schools was attended by teams of three people from each school. The school team included the principal and two mathematics teachers. During this initial meeting, school teams developed project proposals for spring semester site based staff development. These site proposals were shared with the other participating schools and staff members, and then revised. Staff members facilitated the six month school designed programs which were conducted from January through June, 1996. At the end of the spring semester, staff members who had worked with the programs conducted in-depth individual interviews with mathematics department chairs and, in ten out of the schools, they also interviewed the principals. The main purposes of the interviews were to determine what actually happened and to ascertain both the department chairs' and principals' perceptions of the impact of the staff development on their teachers.

particularly on teachers' attitudes, pedagogy, and on the mathematics curriculum being used in their schools. The staff members who conducted the interviews included one high school principal, two district mathematics coordinators, one mathematics department chair, and a mathematics education faculty member at the local state university. Interviews were not taped at the request of the subjects, but notes were taken and summaries were written for each of the interviews.

Subjects. The 19 schools were from three different districts. Two of the districts are high school districts and one is a K - 12 district. The schools range in size from 1200 to 4000 students, with a minority representation of 30% to 80%. All of these schools are comprehensive schools serving grades 9-12. Data include a total of 19 proposals along with interviews from 19 department chairs and 10 principals from the 19 schools.

Questions. The questions below were designed to ellicit information on each school's plan, the plan's goals, the materials acquired and how well they related to the plan's goals, and the participants' perceptions of the value of the opportunity presented by the project as well as the success of their own plan. A basic set of questions in each of these categories was asked of all subjects, with followup questions varying in order to clarify responses as necessary.

- Describe what your staff holds to be the goal or goals of mathematics reform. Are there different viewpoints?
- What was the goal(s) of your project? What were you attempting to do?
- What kinds of in-service experiences did you have? Did you have meetings? If so, how were they structured? (all-day, after school) Who planned and facilitated them? What was the goal(s) of any such meetings?
- How do you believe your staff felt about having the opportunity for these in-services and/or meetings? In your opinion, did they feel any differently after such experiences than before?
- Comment on any change in attitude toward equity in access to mathematics that you noticed in any one of your mathematics staff as a result of any of these activities.
- What materials did you choose to acquire with the funding provided? How does that acquisition of these materials help you toward the goals you mentioned earlier?
- With respect to that "next step", however small, in implementing change in your mathematics program, where do you believe you are now that you would not have been without the opportunities that this funding has helped to provide? Specifically, what obstacles did this funding help you to step over or what gaps did it help you to bridge?
- What other sources of funding were available to you to accomplish the same goals?
- What if any effect do you believe this program may have had on the level of administrative knowledge and support for the mathematics program at your site?
- Given the same amount of funding, what other kinds of things might this grant have done for your program that would also have been helpful or, perhaps, more helpful?
- What is the **next step** now for your site's math program and/or staff?
- With your "next step" in mind, what kinds of support do you believe will be crucial to success at your site?

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Results

The project proposals and responses to the interview questions were examined for patterns; a selection of the results will be discussed here in the categories outlined in the questions above.

All schools reported perceptions of the goals of mathematics reform to be the movement toward new techniques of presentation and delivery of content, as well as of assessment, aimed at opening up access to mathematics for all students. Basic to the goals was the presumption that all students will need such knowledge and skills in the workplace of the future, whether their job requires a college education or not.

All schools reported the perception that this reform effort stresses more problem solving and less symbolic manipulations, and more group work or cooperative learning, using activities with real world applications and integrating technology as a tool for modelling. Department chairs reported that this project did not change any perceptions with respect to the goals of mathematics reform, but it did help departments to clarify and to articulate those perceptions, helping to bring individual thinking on this issue closer together.

All of the schools will have an algebra-for-all for 9th graders in the fall of 1996. The inservices designed to address these programs varied both in goals and emphasis. Basically, the schools could be placed into three categories on the basis of their mathematics curriculum and textbook in existence during the academic year 1995 - 1996.

The first category of schools is the set of schools which had implemented a course one in an algebra-for-all program for 9th graders. Six schools, at least one from each district, were working with a common mathematics program for all 9th graders. For these schools, the site-based program included work on the second course, development of student and parent support programs, work with technology, and discussion of assessment and minimum requirements needed to continue to subsequent mathematics courses.

Category two schools currently teach a traditional algebra-for-all program. Of the four schools comprising this category, one school focused on the selection of an integrated text for the 1996-1997 school year; another worked to develop units which will help with the transition; a third used the institute to align various funded activities; and the fourth school used the time for " attitude adjustment". Two of these schools are moving to a more innovative curriculum next year.

The nine schools in the third category are using several different textbooks including classes taught from traditional books as well as sections of algebra/ course 1 taught in experimental and integrated programs. In two schools there exist four different algebra 1/course 1 programs. The multiple curricula have allowed for experimentation but, at the same time have reduced the communication across the faculty. Similar experimentation will take place through next year in at least four of the schools. Inservice activities included developing an understanding of mathematics reform, standardization of course expectations, liaison with middle schools and work with technology.

In most of the projects, the department chairs functioned as the project director. More than two thirds of the chairs reported that the institutes were beneficial and that their objectives had been accomplished. A common summary comment was that the site based meetings allowed everyone to feel as though there were now more teachers and administrators "on the same page".

Comments concerning the next steps varied as a function of the particular institute and school curriculum. Most of the department chairs believed that the next step for their schools involved more staff development opportunities, especially time to meet and to plan together, to meet with feeder school staff, to see classroom demonstrations of new curriculum and to work with others doing the same things at other schools. More work with technology such as graphing calculators, computer software, and CBLs was mentioned by almost all.

There were a number of areas of concern expressed by the principals as they reflected on next steps needed to continue the move toward reform in mathematics. While the "Building Bridges" program had put particular stress on implementation of algebra I/course 1, the principals saw a real need to provide continued staff development to support changes, especially the broadening of instructional strategies and the integration of technology. Three principals thought there was a need for closer articulation with feeder schools, especially with respect to the coordination of mathematics curriculum. Interdisciplinary strategies were raised by four other principals as the next step in making mathematics connections across the curriculum. Three principals were concerned about the failure rate and the need to continue working on ways to support the C/D students. Given the same amount of funding, none of those interviewed would have spent any differently. What they need for the future is on-going support.

Involvement of the principals from the conception of the institutes was valued. Department chairs felt strongly that it was helpful to have had site level administrators at the planning meetings. They felt it added weight to the plans, a sense of seriousness to the planning sessions, a sense that someone really does care and a sense of need for follow through on the part of their department members. They felt it helped for all to be aware that administrators knew what the plan was to achieve results. As the principal and then the department chair were interviewed, there was an implicit but perceivable feeling of a team working towards a common goal.

Conclusions

The institute designs, reflections of the principals and the observations of the department chairs give us a glimpse of what a project supporting mathematics reform might need to include. It is clear that both the department chairs and principals felt that such an effort must continue to support teacher risk-takers, concentrate on continued training in instructional strategies and the infusion of technology, meet with feeder schools and help practitioners structure an environment that seeks to include every learner and value diversity.

One of the most strongly held beliefs about the value of this project, in the minds of those interviewed, was that it got their math faculty talking with each other about the education of their students. This "educational conversation" produced, they felt, more cohesive departments, a more common focus, more shared values and a more effective approach to mathematics education. It may now be time to *extend the educational conversation* to others who, in turn, can help to expand, support, sustain and refine the mathematics reform effort. In the opinions of the principals and department chairs interviewed, among already proven values of Equity 2000 has been its ability to engender just such conversations.

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THE COMBINED INFLUENCES OF INNOVATIVE CURRICULUM AND A PROFESSIONAL DEVELOPMENT PROGRAM ON TEACHER CHANGE

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This report describes two case studies concerned with the combined effects of innovative curriculum and a professional development program on teacher change. Teachers who might otherwise have been expected to have difficulty implementing reform recommendations were able to make substantial and significant changes in their actual classroom practices. The interplay between professional development and curriculum involved a cycle of providing the rationale for new content and pedagogy and connecting it explicitly to the lessons in the curriculum for each major mathematical topic. The combination produced different results for different teachers but in both cases brought teachers' instructional practices closer to the goals set by the NCTM standards.

The standards set by the National Council of Teachers of Mathematics (1989, 1991, 1994) articulate a national vision of what constitutes reformed mathematics teaching and learning. The realization of this vision demands that all teachers engage in reformed practices. While there are many teachers who are both willing and able to make changes in what mathematics they teach and how they teach it, there are others who are uninterested or unable to independently make changes in pedagogy and in the mathematics content that they stress in their classrooms. How to bring instructional practice into alignment with reform recommendations in all classrooms in a school or system is a major dilemma for educators.

One approach to initiating change is for school systems to adopt curricular materials that support reform (Tierney & Corwin, 1993): innovative textbooks provide teachers with lessons that stress a broad range of mathematical topics and use methodology that situates learning while engaging students. Lambdin and Preston (1995a) have used caricatures to classify teachers' responses to implementing innovative curricula. Subsequent research on their part (Lambdin & Preston, 1995b) questions whether or not it is possible for all teachers to implement curricular reform. This report describes two case studies of the effects of the adoption of an innovative textbook series in conjunction with a substantial system-wide professional development program on teachers' ability to bring their practice into alignment with reform recommendations.

The Project

This report documents one aspect of a multi-year study of the effects of an extensive professional development program in mathematics on teachers of grades PreK-12 in an urban school system. The professional development program uses a model that combines and integrates instruction in mathematics content and pedagogy, and introduces and supports mechanisms that contribute to the implementation of content and pedagogical changes at the classroom level. Support services that encourage shared risk-taking are made available to teachers, who can elect to enroll in short and long courses and workshops, and/or meet regularly (weekly, bi-weekly, monthly) with a mathematics lead-teacher. The mathematics lead-teacher provides individualized, classroom-based support for a teacher by modeling lessons, co-teaching, planning with the teacher, or posing questions about instruction, assessment, or individual students.

The textbook series used in grades K-3 is *Moving Into Math* (Mimosa, 1992). *Moving Into Math* is a language-based, problem-centered mathematics program. The content and instructional approach of this curriculum are aligned with NCTM recommendations. This series was first used in the school system in 1994-5.

Methodology

Two first grade teachers are the subjects of this research report. Both teachers participated in a professional development course during the year in which the *Moving Into Math* series was introduced. This course, designed for teachers of grades K-2, focused on assisting teachers in understanding key mathematical concepts, and in understanding how young children construct knowledge about these concepts.

The case studies were developed over a period of one year using quantitative and qualitative data from the following six primary sources.

- Field notes on observations of the teachers during course meetings (16) and during their individual meetings with lead teachers (over 30 for each teacher).
- Reflective essays and reports that the teachers wrote for the course (10 each).
- Interviews with the teachers (3 each).
- Teacher surveys on attitudes, beliefs, and content and pedagogical knowledge.
- Notes on informal conversations with the teachers.
- Observations of the teachers teaching mathematics (15^+ observations each).

Nancy, a teaching veteran of 16 years, became involved with the professional development program in 1993 because she was very anxious about her own understanding of mathematics. She thought that taking another course might help her implement the *Moving Into Math* series. Nancy had an opportunity to examine the new series during the adoption process. She was excited about "having a series that would help [her] put into place recommendations from NCTM." At the start of the case study, teaching observations and survey results indicated that Nancy was a fairly conventional teacher. Even though children in her classroom were grouped to work together, most of her lessons were teacher-directed, and involved telling children how to think about a particular mathematics topic. She occasionally used manipulative materials or deviated from her basal textbook to create a unit of study. Nancy used alternative curriculum materials, such as lessons by Marilyn Burns, to guide her selection of the activities for these units. She was intellectually open to change, but had not been very successful in actually incorporating meaningful changes into her practice.

Louisa became involved in the professional development program in order to accumulate professional development credits to keep her elementary certification current. She had not taken a course in mathematics or mathematics education in over 20 years. At the start of the case study, observations of Louisa's classroom and survey results indicated that she was extremely traditional: children were seated in rows, and were not allowed to speak unless called upon. Her teaching methodology stressed rote and repetition, and her lessons were worksheet driven. Louisa believed that a student having difficulty in mathematics should be subjected to even more practice on skills. Louisa was extremely anxious about using the new textbook series, and would certainly not have been using it except for a mandate from the principal of her school. She stated in one of her written reflections that "I resent the implication that after 21 years of teaching I don't know how to teach mathematics."

Observations

This report focuses on one aspect of the interplay between the professional development program and teachers' use of a curriculum: How does teachers' knowledge of key mathematical ideas, and how young children construct knowledge about these concepts, affect their interpretation and use of the curriculum?

At the beginning of the school year both teachers were observed teaching the same lesson. The textbook suggested that students, working together, explore all the combinations of two numbers that sum to eleven by separating eleven objects into two groups. Students were then to record, either with symbols or pictures, the number in each group, on a sheet of paper with an "11" written at the top. Following this activity, the textbook suggested that teachers have students discuss both their strategies for finding combinations and the patterns they observed.

Louisa adapted the lesson to fit her interpretation of its purpose, and her beliefs and attitudes about teaching and learning. She modeled how you could separate eleven counters into two piles, and then showed students how to record the amounts in each pile on the paper provided. She paired students and sent them off to work together, but she failed to mention that they were to try to find all combinations. While the students worked, Louisa was primarily concerned with discipline and noise. She constantly reminded students to work quietly and she rarely interacted with students about the mathematics. She also omitted the discussion; instead, she called the class together and asked four groups to each share one combination they found. She wrote the combinations (e.g., 1 and 10) on the board , added ones that were not provided by the students, and then had the class recite together the combinations listed as if they were reciting addition facts (e.g., "1 plus 10 equals 11"). Her belief that drill and practice are the essential learning activities was highlighted by her inclusion of the group recitation.

Nancy, on the other hand, followed the instructional suggestions in the textbook more closely. She interacted with her students while they were finding combinations for eleven, giving hints or suggestions, but she did not ask questions that might require a student to generalize. Nancy did facilitate a discussion during which she encouraged all students to share their findings. She wrote all the combinations they mentioned on the board and asked the students to decide if they had all possibilities. Despite her intuitively motivated question about the number of possible combinations, Nancy did not ask other questions that indicated she had a clear understanding of the mathematical purpose of the activity. For example, she was amazed when her students noticed that as one number in the combination increased the other number decreased, but she did not pursue this line of inquiry.

During a discussion of the lesson with the researcher, Louisa was concerned about the methodology used and the resulting pace of the curriculum. She had not read the background information in the textbook, which might have provided some insights into the role of discussion. Nancy was much more comfortable with the idea of having discussions about mathematics, but felt inadequate when it came to actually facilitating discussions. She wanted feedback on her observation and suggestions for improvement. Both teachers did not fully understand the point of the instructional activity — the activity seemed to them to take a long time for what resulted in a minimal amount of practice on some basic facts.

The *Moving Into Math* series places great emphasis on number relationships. In the professional development course in the weeks following the observed lessons, teachers learned about the development of number concepts and number sense. A variety of techniques were used with the teachers to explore the mathematics and related pedagogy: lectures, discussions, and activities; readings of articles and book chapters on how children construct meaning; discussions about students' actions and insights; and brainstorming and role playing of student-teacher interactions and questions. Teachers were also asked to identify instructional activities from the textbook that assisted students in constructing different number relationships such as part-part-whole. Since the curriculum was new to the teachers, examples of how more sophisticated number concepts were related to earlier activities were presented by project staff. This helped teachers to grasp how distinct lessons on, for example, "benchmarks of ten" and "part-part-whole relations" might fit together when students explored the concept of grouping. Lead teachers, in their weekly meeting with teachers, asked them to reflect on the activities in their textbook, and their own responses to the activities in the context of the discussions of learning and teaching that were occurring in the course.

Nancy and Louisa reacted in similar ways to understanding more about number concepts and how these ideas were developed in the curriculum. They stopped questioning the purpose of number sense activities, and instead regularly commented that they understood why these activities were included at this grade level. They subsequently spent more nd more instructional time on number sense activities. Nancy mentioned that she felt more comfortable leading discussions. Louisa did not immediately start facilitating discussions, but she did make sure that all of her students shared their solution strategies. Most importantly, each teacher began to structure her lessons so that students had to actively construct meaning from the instructional activity, and both emphasized the importance of students' explanations and reasoning.

Students' reactions and responses to the instruction provided positive reinforcement to the teachers that their students were learning. This in combination with the professional development activities seemed to provide overwhelming evidence to the teachers that both the content and the methods used were educationally sound. With time both teachers asked their students questions and facilitated discussions that illuminated the underlying ideas in an activity. They had students discovering properties such as commutativity and conjecturing about patterns, and then discussing, writing about, and applying their ideas and understanding. Louisa stopped inserting the additional drill and practice activities altogether following a class discussion about sending mixed messages about what is valued. They both transfered more and more authority to individual students and their class as a whole for determining the correctness or reasonableness of a response.

Although by the end of the year Nancy and Louisa were both working in "changed classrooms," there were subtle and profound differences between them in implementation of the new curriculum. Louisa followed the suggestions in the textbook more closely than earlier in the year because she understood their purpose. Once she became convinced that the content and methodology of the lessons were enabling students to learn more, however, she seemed to abdicate authority for decision-making to the curriculum: she did not deviate from the order or emphasis of lessons in the textbook, and at times she appeared to be rushing through lessons in order to move on to the next topic. Nancy, on the other hand, started to analyze individual students' responses to her questions; she then adapted the instruction to reflect her understanding of their knowledge or misconceptions. For example, when discussing combinations for the number 16, Nancy's students again observed that as one number increases the other decreases. The next day Nancy asked students if this increasing/decreasing pattern would hold for any number. Two days of lessons involving data collection and generalizations ensued.

Conclusions

Nancy is on her way to embracing what Schifter and Fosnot (1993) call a "paradigm shift in what it means to understand and teach mathematics." Louisa's practice, on the other hand, is full of contradictions. Some aspects of her changed practice indicate a reformed outlook on learning, but others suggest that she has simply replaced one set of prescriptions with another. However, compared to the mathematical experiences her students had in previous years, there is no question that they are "better served" by the instruction that they are receiving today.

The combination of innovative curriculum emplaced and sustained by a coordinated program of professional development led these teachers to significantly change their classroom practices in a fairly short period of time. The interplay between professional development and curriculum involved a cycle of providing the rationale for new content and pedagogy and connecting it explicitly to the activities in the curriculum for each and every major mathematical topic. As teachers gained greater knowledge about mathematics and learning mathematics and received verification that their reformed practices were making a difference in students' learning outcomes, they more willingly considered the philosophy and methods that were being discussed and suggested in the professional development program. The professional development program enabled teachers to examine the interrelated factors that contribute to changed practice — their students' learning, their own learning, and their belief in and promotion of new approaches to teaching mathematics. Using an innovative textbook series in conjunction with professional development that stresses learning and cognition assisted these teachers, who might otherwise have rejected or failed at reform, to make substantial and meaningful changes.

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REFORM IN PRACTICE: THE INTERSECTION OF PROGRAM GOALS AND PARTICIPANTS' EXPECTATIONS[1]

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This paper reports on the intersection between the goals of a mathematics teacher enhancement program and its participants' expectations. The authors seek to understand the dynamics of this interplay. In doing so, they find that there are three factors of emotive orientation influencing participants' expectations of the program--visions of reform, confidence (both as a teacher and as a learner of mathematics), and constraints (intrinsic and extrinsic). Consequently, they find that participants' reactions tell as much about the program as they do about the individuals as learners as well as their perceptions of reform.

Attempts at reform in mathematics education have been many and have documented well the problematic nature of innovation (e.g., Burke & Littleton, 1995; Lambdin & Preston, 1995; Weissglass, 1994). Educational reform, it is suggested, is problematic for several reasons. Not only does it involve professional development, technical expertise, and content knowledge, but it entails personal change. By and large, research on teacher change has focused on teachers in their classrooms through a series of narratives such as cases or composites (Lambdin & Preston, 1995; Schifter, 1994; Sztain & Lester, 1994). However, few studies give us a behind the scenes look at the inner workings of the professional development programs in which these teachers participate. At most, we get a brief description of the outline of the program and learn little of the staff's interpretation and consequent implementation of goals. This paper attempts to fill the gap by focusing on the often enigmatic intersection between the goals of a teacher enhancement program (as perceived by the program staff) and participants' expectations.

Background

This paper is part of the research on a teacher enhancement program directed at promoting mathematical knowledge and leadership skills of three groups of 30 teachers each, of grades 3-8. Over the course of three years, each group of participants attends six weeks of summer institutes, three the first summer, two the second, and one the third.

During the summer institutes, teachers take courses in geometry, numbers and algebra, and probability and statistics. They also take one course on the research on learning mathematics and another on the implications of reform for their own classrooms and schools. Participants sit in small groups and discussion and sharing of ideas is encouraged in all the courses. Technology, manipulative materials, and a variety of exemplary curriculum resources are integrated throughout the courses. During the school year, support mechanisms are in place in the form of four one-day inservices, monthly workshops, and regular visits (for purposes of instruction, modeling, consulting on planning lessons, or technology use in the classroom) by program staff to participants' classrooms.

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[1] This research is supported in part by the National Science Foundation #ESI-9253845. The views

Method

To investigate the intersection between goals of the program (as interpreted by various staff members) and participants' expectations, we collected data from a variety of sources: observations during summer institutes and at participants' schools, interviews of participants and program staff, participants' journals, statements of interest, and surveys. In order to understand the impact of the program, we are developing case studies of select participants. Our analysis focuses on three factors of emotive orientation affecting the participants' expectations about the program--visions of reform, confidence (both as a teacher and as a learner of mathematics), and constraints (intrinsic and extrinsic). In this paper we first present some of the staff's interpretations of the goals of the program. We then share excerpts from three cases of teachers to illustrate the interplay between their expectations and the program's goals.

Findings

Program Goals

Interviews and observations of the program staff suggest a wide range of interpretations of the goals of the program. These interpretations are further reflected in the implementation of the courses and in the interactions between individual staff and participants. Everyone on the staff appears to agree that in the three mathematical content courses, participants are there as learners of mathematics, and in the other two courses, they are there as teachers. However, how this is translated in the practice varies considerably. For example, Tom, who has been a mathematics professor for over twenty years, focuses on the role of participants as solely adult learners of mathematics and presents participants with challenging mathematical problems. He wants them to see mathematics as "an exciting field where things do hang together," as he explains, "this little piece over here actually is connected to this little piece over here." Tom's teaching style conveys his own sense of playfulness and enjoyment with mathematics, and most participants comment positively on his personable approach. Yet, mathematically, this course is frustrating for many participants who find the content difficult and irrelevant to their teaching.

Ellen focuses on the role of participants as classroom teachers and presents them with activities from innovative mathematics curricula for grades 3-8. In her professional career, Ellen has come full circle, she has been a classroom teacher, a professor of mathematics education, and now a district curriculum specialist. Thus, she is aware of how little time teachers have to look for good mathematical activities. Ellen also focuses on modeling a

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expressed here are those of the authors and do not necessarily reflect the views of the funding agency.

teaching approach that mirrors the call for reform. Overall, Ellen's course is not as mathematically demanding as Tom's, but the participants report that they enjoy it immensely and find it relevant to their teaching.

The third mathematical content instructor, Dave (a professor of mathematics), presents the participants with a careful, in-depth exploration of concepts in a content area with which most participants have had little experience either as teachers or as learners. Those teachers who have previously tried to teach some of these concepts appreciate the very thorough discussions on the topic. However, those others who have previously learned the material in an algorithmic manner, object to the slow pace of this course. Judging by the participants' journals, as well as subsequent observations of their teaching, this instructor is quite successful in accomplishing one of his articulated goals, "I hope to be able to model the process that maybe they will be able to use with their students."

The differences in interpretation and implementation of the program goals are not limited to the mathematical content instructors[2] alone. In fact, one co-teacher commented as part of her evaluation of the summer component: "There seem to be different opinions among professors, co-teachers and participants as to the intent of the institute," as she proceeded to illustrate the different approaches in the courses. She then concluded by writing, "Since each class is *so* different, I feel there is frustration among many. Maybe these differences should be discussed at the beginning with new participants so the frustration level could be reduced." Similarly, a participant wrote in her journal during the second summer: "I am sensing my frustration about the lack of a real clear philosophy from this program. I am used to being in an environment where everyone may not agree with it but there is a very strong solid philosophy. Maybe where there is none is OK, but I'll have to think more about it." This participant is not alone in feeling this way. Several of the participants, mostly among those who had had more exposure to reform oriented professional development programs, corroborated this need for a clear philosophy in their journals.

> I think that at some point I would like clarification of what [the program]'s philosophy is on methodology and materials. Exactly, what kind of reform is [the program] proposing? (Betty, Journal)

Betty's observation about methodology and materials is not surprising, for the program does not espouse any one curricular material, nor does it promote its own pre-fabricated materials. Was there no philosophy? Or were there different interpretations about the call for reform and were these reflected in the implementation of the program? Those in charge of teacher enhancement programs often adapt (rather than adopt) the call for reform just as teachers do (Lambdin & Preston, 1995). These different interpretations are likely to promote already varying orientations to reform among the participants, as the three cases illustrate. As an aside, there are understandably many marked differences among participants. However, we sense a pattern emerging, wherein participants are noticeably shaped by district orientations and their professional staff development.

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[2] Due to space restrictions, we necessarily limit our discussion in this paper to these three staff members.

Participants' Expectations

Alyson: "I am feeling like a reform in math education drop in the ocean of educational unawareness." Alyson is very knowledgeable about the reform literature, espouses its rhetoric, and is perceived as a leader in mathematics education reform by her school district. She is a confident teacher and her job promotes and facilitates her flexibility to try out innovations. As a learner of mathematics, Alyson acknowledges that she has a long way to go, which was her primary reason for participating in the program. However, Alyson's adherence to what she calls "teaching for meaning" and other aspects of her well-defined vision of reform clash with her expectations about the program and, at times, contradict her own role as a learner. Alyson thrives on Dave's reflective, in-depth approach to teaching and this is subsequently reflected in her own teaching. On the other hand, with regard to Tom's course she writes:

> I have gone through more "emotional stages" during these lessons, because of the style of the class. In my current professional work I have inundated myself with the philosophy of teaching for meaning. It always feels odd to me now to be taught in a way in which the meaning is not valued or emphasized....Different ways in which people think are not honored....I have not seen more than one solution to any problem on the board. (Alyson, Journal)

Sharing approaches to problems is very important to Alyson, as she makes a similar comment about Ellen's class. In that course she is not concerned about the level of the mathematics, however, but she does express some disappointment at what she perceives as a focus on getting answers to the activities, rather than a discussion on the process.

Susan: "I'm getting that 'Aha!' experience and I'm loving it and I hope that I can pass that on to my kids." Susan particularly enjoys Tom's class because she likes to have her thinking stretched in the same way she stretches her students' thinking. For the last three years, Susan has been teaching mathematics to the "top" third grade students at her school. This position gives her considerable flexibility in what she chooses to do with her students. While she is bound to make sure that students meet the basic grade level objectives, the students do so quickly and easily, which allows Susan to extend those concepts or teach new ones. Hence, she has few curricular constraints on her teaching. Susan's vision of reform revolves around a switch in focus from a textbook driven curriculum to one that is meaningful to students, rich with applications to their everyday lives, and driven by their interests. Susan believes that in order for teachers to implement this vision they need a strong content knowledge, sound pedagogy, support from the administration, and knowledge of what children can do at a particular grade level. In her application to the program, Susan writes:

> Children are at the heart of the decisions made [in my school and district], and if I am able to enhance student's appreciation of math through the ideas of mathematics reform presented in the program, then I would consider myself a successful teacher. (Susan, Statement of Interest)

Susan appears highly confident both as a teacher and as a learner. As a teacher, she eagerly follows the students' lead as they explore and discover "the beauty of math." She is very comfortable with telling her students "I don't know, let's find out." She is always willing to listen to her students as they offer alternative explanations. She is confident that her knowledge of mathematics will enable her to guide the students through their explorations. As a learner, Susan thrives in mathematically challenging situations. For Susan, the greatest impact of the program is that is has enhanced her confidence to follow her students into uncharted territory.

Mary: "I hope to store a wealth of ideas to use in my classroom." Although Mary is interested in obtaining resources, particularly those that illustrate the uses of mathematics in everyday life, she appears to be a discriminating consumer. Despite being a new teacher, Mary is highly confident. She develops her own units by following a theme from another subject, such as biology, and does not follow the textbook. In order to adapt materials to both her students and those themes, Mary believes that she must continue to learn mathematics herself, hence her interest in participating in this program. The program provides activities she can incorporate into her lesson plans, but she does not make the activities become the lesson. She also finds many parallels between her recently completed teacher preparation coursework, which has helped shape her present vision of reform, and some of the approaches to mathematics instruction as modeled in this program. In this sense, the program seems to reinforce Mary's current practice. Another impact of the program on Mary has been the opportunity to share ideas with fellow teachers. This is a recurrent theme in her journal. Mary seems to enjoy every aspect of the program. This is reflected in her agreement that she is "like a sponge" at this point in her career, trying to absorb as much as possible. As for constraints, Mary teaches in a district with a very traditional orientation (focus on the basics), where she must be prepared to justify her approach to parents and administrators alike.

Conclusion

A program of reform is not successful on its organization alone, for it continually falls under scrutiny as evidenced in the often reiterated comment along the lines of "it's not what I expected." To understand what this statement means, we find ourselves seeking to establish the significance of the match between program goals and participants' expectations. The intersection of participants' expectations with a teacher enhancement program can elicit, on the very same topic, as it has here, reactions at polar extremes (e.g., "I learned a great deal."-or-"This had no relevance at all to me."). Yet, the validity of concerns raised by participants is neither unwarranted nor wholly attributable to the program alone.

Individual orientations to personal change encompass internal and external factors which may or may not go unnoticed by the individual. The confidence in one's abilities as a learner and professional, and the constraints on practice influence heavily the vision of reform. Intersected with a program of reform, participants' expectations may take on a life of their own in the change of classroom teaching practices, or they may fall into the already brimming pool of curricular demands and reform movements. A perfect match is difficult to achieve and is again heavily influenced by individual beliefs (of the way we learn, the content we perceive appropriate or correct, and our own personal quirks) and by goals as perceived and implemented by individual instructors.

To what extent is Alyson's prior professional development affecting the nature of her involvement in this program? Alyson may very well remain the same, we sense. Her theoretical framework in regards to pedagogy and professional development seems to have been established prior to the program, such that it may be difficult for her to look outside its lenses. With regard to Susan, we have asked ourselves if she would have had different expectations had she not had the freedom that she now has in her teaching position. Certainly this was a boon to her teaching, however, we suspect that her integral vision of reform would carry to other classroom circumstances. As for Mary, as long as she feels that she has a certain degree of freedom in her teaching, she will very likely continue to be an actively involved learner and teacher. In part, we attribute this attitude to her having had the benefit of a teacher preparation program along the lines of the current reform in mathematics.

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THE TEACHER AS A CONSTRUCTOR OF PEDAGOGICAL EVENTS ABOUT CONTENT CURRICULUM EVENTS: TOWARD AN EMPIRICAL VALIDATION OF A THEORETICAL FRAMEWORK INTEGRATING CURRICULUM, TEACHING AND LEARNING IN STUDIES OF LEARNING TO TEACH

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In this paper I propose a theoretical framework for examining the process of learning to teach specific mathematical content. The focus is on the process of transforming content curricular events (curriculum) into pedagogical events (teaching) that are constructed with the purpose of facilitating students' construction of mathematical knowledge (learning). Three pedagogical events are described: explanations, representations, and questions. Some factors that are considered to play a critical role in teachers' construction of these pedagogical events are: mathematical content knowledge, pedagogical content knowledge, curricular content knowledge, conceptions of teaching, learning, and the nature of mathematics and students' cognitions and affective factors when learning the intended mathematical content curriculum events. I also describe some findings that add some empirical validation to the model.

Understanding the process of learning to teach mathematics is a fundamental problem of the psychology of mathematics teacher education just, as understanding the process of learning mathematics is a fundamental problem of the psychology of mathematics education. Understanding the process of learning to teach has potential implications for redesigning teacher education programs so that teachers learn to construct environments pedagogically rich so that students learn the intended mathematical knowledge as envisioned in the current reform efforts (National Council of Teachers of Mathematics, 1989). Research can enhance our understanding of this process. However, a review of the literature reveals that relatively little attention has been paid to understanding this process. For example, Brown, Cooney, and Jones (1990) focused on philosophical issues in mathematics teacher education. In contrast, Brown and Borko (1992) reviewed research related to becoming a teacher not only from a specific mathematics focus but also from a generic perspective. However, as stated by Brown and Borko, most of the research reviewed describes teachers' beliefs, knowledge, and cognitions without examining the relationship between those constructs and teachers' instruction. In addition, little research attention has been devoted to examining curriculum and teaching in an integrated way (Doyle, 1992). Curriculum and teaching are inherently interwoven. As stated by Doyle (1992), "a curriculum is intended to frame or guide teaching practice and cannot be achieved except during acts of teaching. Similarly, teaching is always about something so it cannot escape curriculum" (p. 486). On the other hand, the most important purpose of teaching is to facilitate students' construction of mathematical knowledge (learning). A question arises: why is it that little research has been conducted on learning to teach when one, if not the most important, goal of the whole teacher education enterprise is to help teachers learn how to teach? Cruickshank (1990) suggests that one reason may be the lack of models to guide inquiry in teaching, in this case, in learning to teach mathematics. We

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have some models that can help us to guide inquiry in teaching (e.g., Wilson, Shulman, & Richert, 1987), but we do not have models to guide inquiry in learning to teach that integrate curriculum, teaching, and learning. The purpose of this paper is twofold: to propose a theoretical framework that integrates curriculum, teaching, and learning in research on learning to teach, and to describe some research findings that will add some empirical validation to the theoretical framework.

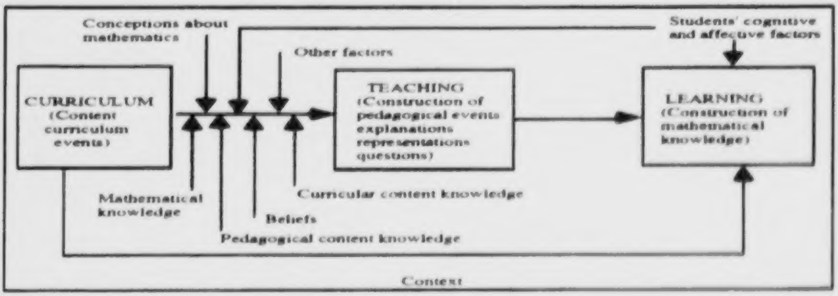
The Theoretical Framework

Learning to teach is considered to be a lifelong process (Brown & Borko, 1992). This process begins when a future teacher, as student, observes teachers teaching and experiences what Lortie (1975) termed *apprenticeship of observation*. Prospective teachers learn to teach from teacher education programs. Finally, teachers continue to learn to teach from the practice of *teaching*. During these three phases teachers acquire and develop the knowledge and pedagogical skills needed to teach mathematics. The theoretical framework proposed here focuses on the teaching act and its links between curriculum and learning.

I conceive of a curriculum as made up of content curriculum events. A mathematical content curriculum event is each mathematical idea or object (e.g., concepts, formulas, theorems, axioms, algorithms or procedures, etc.) identified in the curriculum or in a curriculum text such as curriculum guides or textbooks.

Shulman and his associates (Shulman, 1986; Wilson, Shulman, and Richert, 1987) conceptualize teaching as a process of transforming personal understanding of content knowledge to make it comprehensible to students. That is, teaching is a pedagogical process. Doyle (1992), on the other hand, describes teaching as both a pedagogical and curricular process of transformation because a curriculum cannot be achieved without acts of teaching and teaching is always about something. That is, Doyle (1992) conceptualizes teaching as the process of transforming the curriculum. I contend that teachers do not transform (teach) a curriculum in a general way but, rather, they transform specific pieces of the curriculum that I call content curriculum events using three main types of pedagogical events: explanations, representations, and questions. An explanation is described by Leinhardt, Putnam, Stein & Baxter (1991) as "an activity in which teachers communicate subject-matter ... [it] is not only what a teacher says or shows to the student, but also includes the systematic arrangement of experiences so that the student can construct a meaningful understanding of a concept or procedure" (p. 89). These scholars define representations as "physical or conceptual objects or systems of objects that embody mathematical entities or ideas" (p. 89). A question is a formulation that calls for an answer. The purpose of most questions are to monitor, check, and assess students' learning. I conceptualize teaching as the construction of pedagogical events about content curriculum events (curriculum) in order to help students construct the mathematical knowledge embodied in the content curriculum events (learning). Learning to teach can then be defined as the process of learning to construct these pedagogical events.

What are the factors that influence teachers' construction of pedagogical events about content curriculum events? The first factor that seems to be evident is teachers' content knowledge. The importance of this type of knowledge is supported by several research studies that have examined teachers' knowledge about specific mathematical content (e.g., Ball, 1990; Even, 1993; Fennema and Franke, 1992; Simon, 1993; Tirosh & Graeber, 1989; Wilson, 1994) or that have investigated how teachers use their mathematical knowledge in teaching situations (e.g., Ball, 1991; Fennema & Franke, 1992). Others factors are teachers' conceptions about the nature of mathematics (Dossey, 1992), teachers' beliefs about teaching and learning (Thompson, 1992), and other affective factors such as attitudes and emotions (McLeod, 1992). In addition to content knowledge, Shulman and his colleagues (Shulman, 1986; Wilson et al., 1987) argue that teachers draw on pedagogical content knowledge and curriculum content knowledge during the process of transformation. Regarding learning, four factors seem to contribute to how students construct mathematical knowledge: students' cognitions, teaching, and the content itself (curriculum) (Kieran, 1992) and students' affective domain (McLeod, 1992). Finally, the curriculum, teaching, and learning are embedded in a particular context, culture, or situation that influences teaching and particularly what students learn (Brown, Collins, & Duguid, 1989; Doyle, 1992). Depicted below is the theoretical framework that I put forth for examination, discussion, and to stimulate further research on learning to teach.


Conceptions about mathematics
Students' cognitive and affective factors
Other factors
CURRICULUM (Content curriculum events)
TEACHING (Construction of pedagogical events explanations representations questions)
LEARNING (Construction of mathematical knowledge)
Mathematical knowledge
Curricular content knowledge
Beliefs
Pedagogical content knowledge
Context

The Empirical Validation

I am using the data collected for another research project to provide some empirical validation to the theoretical framework (Contreras, in progress). I will use the following content curriculum event to illustrate some categories of the framework and relationships

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among categories: the algebraic definition of division ($\frac{a}{b} = a \cdot \frac{1}{b}, b \neq 0$). When Mr. Kantor, the participant of the larger project, was faced with teaching this content curriculum event, he constructed three story-problems or physical situations to illustrate why $\frac{a}{b} = a \cdot \frac{1}{b}$. He used three numerical examples: $\frac{7}{2}$, $5 \div \frac{1}{2}$, $21 \div \frac{3}{4}$. He asked students to construct a story-problem representation whose solution can be represented by 7/2. A student said "seven pieces of pizza divided by two people." Then Mr. Kantor said that the answer to 7/2 and 7(1/2) was 3.5 and asked students to explain why that is. One student said "because multiplication is the opposite of division. One half is the opposite of two." Mr. Kantor did not accept that answer as correct and then he constructed an explanation using the model *of* for multiplication: "if you have seven pieces of pizza and each person gets half *of* that, that's seven times one half." Since students were not very successful in constructing the explanation requested by Mr. Kantor, he used another example, 5÷(1/2). In this case students had trouble in constructing a story problem representing 5÷ (1/2). After several attempts, the students, helped by Mr. Kantor, created the problem "You have five dollars. You wanna find how many pencils you can buy that cost fifty cents a piece." When Mr. Kantor asked students to explain why the solution to this problem can be represented by 5·2, they provided the following answers: because it is the opposite, "because it equals ten," "because when you divide ... you can do the same thing by multiplying by the reciprocal," "because it is," etc. Mr. Kantor explained why 5÷(1/2) = 5·2 in the following terms using a different situation: "I wanna give each kid fifty cents. I have five dollar bills, all right? ... I've gotta to convert that. . . . I have to go the bank and get fifty cent pieces for five dollar bills. For every dollar bill they give what? . . . Two fifty cent pieces, right? and so ... five times two is how many fifty cent pieces I get. . . . how many times a half goes into five is the same as five times two." Mr. Kantor constructed another story problem to illustrate why a/b = a(1/b) using 21 ÷ 3/4. Both students and Mr. Kantor faced struggles and difficulties in understanding and explaining, respectively, why 21÷ 3/4 = 21·4/3.

I will state these episodes in terms of the theoretical framework. Mr. Kantor constructed conceptual pedagogical events, explanations, representations, and questions, (teaching) about a content curriculum event, the algebraic definition of division (curriculum), to help students construct conceptual knowledge about that piece of mathematical knowledge (learning). In constructing the pedagogical events he drew on his knowledge of why $\frac{a}{b} = a \cdot \frac{1}{b}$ (mathematical knowledge) using numerical and story-problem representations (pedagogical content knowledge) because he thinks it is important for students to understand the meaning of and connections between mathematical procedures using multiple representation (conceptions about mathematics, teaching, and learning). Students' difficulties and prior knowledge (students' cognitions) as well as students' beliefs, attitudes and emotions (affective factors) prompted Mr. Kantor to construct several examples about the meaning of the definition of division. I did not formally assess students' learning but their responses suggest that both cognitive and affective factors as well as the difficulty of the content might have had an influence on what they learned.

I have provided some empirical evidence to support some aspects of the theoretical model. Even though some researchers have examined some aspects of learning to teach (Borko et. al., 1992; Thompson & Thompson, 1996), I hope that this model will further guide inquiry in learning to teach and that at the same time some results of such inquiry will add empirical validation to the theoretical model or modify it accordingly. The ultimate goal of the research generated by this model is to create content-specific theories of how teachers learn to teach. While I believe that learning to teach specific mathematical content involves both generic and mathematical aspects, the main feature of this model is that it assumes that learning to teach is content specific. In this sense, I agree with the following implication of Marton's (1989) pedagogical theory of content: while we cannot answer the question of how mathematics teachers learn to teach without consideration of the content, we can construct theories of how teachers learn to teach specific mathematical content that may be useful for helping teachers learn to teach *specifically those* pieces of mathematical content.

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SUPPORTING TEACHER DEVELOPMENT THROUGH ELECTRONIC NETWORKS[1]

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This paper explores ways that electronic networks can provide support to elementary teachers reforming their mathematics teaching practice. It examines the content of three message "strings" on three distinct topics, with an eye to the following questions: (1) What are the characteristics of messages which seem to provide opportunities for teachers to rethink or reconsider their understandings of mathematics teaching and learning? (2) How are these messages situated within a string of messages having a beginning, middle, and end? and (3) What is the role of teacher educators or facilitators in extending and deepening the discussion? The paper ends with further questions about how electronic networks might support the mathematics education reform effort.

Introduction

Increasingly, it is being recognized that collaborative communities of reflective practice are an important component of mathematics education reform. These reflective communities can allow teachers to explore together, in new ways, the important mathematical ideas that underlie their mathematics curricula; their own mathematical thinking and the mathematical thinking of their students; and the complex pedagogical decisions that reflect an effort to teach mathematics meaningfully (Hammerman, 1994; Lord, 1994; Nelson and Hammerman, 1996; Schifter, 1996a, 1996b; Schifter and Fosnot, 1993). These communities can be of invaluable support as, over extended periods of time, teachers reconsider what it means to teach and learn mathematics and actually begin to reconstruct their mathematics teaching practice.

Electronic networks hold promise as a tool for cultivating and sustaining these kinds of reflective communities, providing teachers with access to extended professional communities and offering opportunities beyond those available in face-to-face interactions. Network members can participate in substantive discussions at almost any hour of the day; they need not compete with more assertive colleagues who tend to dominate group discussions; they can think through their own thoughts and carefully consider the contributions of colleagues; and a written record remains for further discussion (Harasim, 1990; Riel, 1990).

Projects that have included electronic networks have reported them to be an effective in-service tool. However, much of the analysis of electronic network has consisted of quantifiable features such as numbers of messages, message length, and numbers of

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[11] The authors would like to thank Annette Sassi for her helpful comments on the various drafts of this paper.

participants actively reading and posting messages. Analysis of the actual content of what is shared over these networks, and how this is connected to a reform agenda, is just beginning to be examined by researchers (e.g. DiMauro & Gal, 1994; Honey et al., 1994; Spitzer et. al, 1994).

This paper examines how network discussions can support teachers as they rethink important aspects of their mathematics teaching practice. It focuses on an analysis of three message strings within an electronic network designed as part of the Teachers' Resources Network (TRN).[2] These three message strings, while only a small portion of the exchanges that took place over the network, capture several of the types of electronic interactions that took place.

The TRN electronic network was a private electronic network that connected six groups of approximately ten K–8 teachers, the district's mathematics specialist, and a teacher educator. Each group was located in a different region of the country, and met in biweekly inquiry groups to explore mathematics teaching and learning. The network was created using the FirstClass™ bulletin board system—an extremely user-friendly system even for teachers with little or no prior technology experience—and was used to broaden and extend discussions across inquiry groups.

The Electronic Discussions

The three message strings selected for analysis are the following, along with the rationale for their selection: (1) a message string about a mathematics education resource, chosen because it was an extensive string of 15 messages, all of which failed to dip deeply into issues of mathematics teaching and learning; (2) a message string about moving from concrete to abstract in one's mathematics teaching, chosen because it was a limited string of six messages, involving only four individuals—one teacher, two mathematics specialists, and one mathematics educator—yet served to raise interesting and important issues of mathematics teaching and learning; and (3) a message string about issues of ability grouping for mathematics instruction, chosen because it was an extensive string involving 19 messages and 11 individuals, and also raised interesting and important issues of mathematics teaching and learning. Each is briefly described more fully below.

Message String #1: A Mathematics Education Resource

This message string centered on a recently-published children's book entitled *The Math Curse*. It began when one teacher posted a review of the book, which characterized it as a "neat new book" for "ages > 6 and < 99" and then quoted the inner pocket of the book:

> Did you ever wake up to one of those days where everything is a problem? ... Why? Because you are the victim of a Math Curse. That's why. But don't despair. This is one girl's story of how that curse can be broken.

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[2] The TRN Project was funded by the DeWitt-Wallace Reader's Digest Fund, and ran from the winter of 1993 through the spring of 1996.

The only comment from the teacher was that he had encountered it through another electronic network and wanted to share it. There were four different responses to this initial posting—three from teachers, and one from a teacher educator. The teachers all indicated that they had recently received a copy of the book, had read it to their students, and had found that their students loved it. The teacher educator expressed some mixed feelings about the book—particularly its title—and wondered about the kinds of problems posed in the book. The subsequent postings from teachers and another teacher educator consisted of reports about how much their students had enjoyed the book: "They really got a chuckle out of it . . . "; "They loved it, and wanted to know if I would still have the book next week ... " Messages were short, and none of the messages addressed issues raised by one of the mathematics educators. Nowhere was there a discussion of what students might learn through this resource (other than that "math was everywhere" and could be "fun"), and nowhere were there any connections to teachers' mathematics teaching practice.

Message String #2: Concrete to Abstract

This message string was initiated by a mathematics specialist, and was very brief:

> We're beginning to look at the issue of how teachers help students transition from the concrete activities to symbolically recording that mathematics. Anyone have any ideas?

This message, brief though it was, elicited a series of lengthier responses. The first response was from a teacher, who began by relating her own experiences as an adult learning mathematics through manipulatives —"I loved the way they made my brain crunch when I had to match my abstract understanding to an unusual or surprising model"— and then went on to connect her experiences learning math with the experiences of her students, observing that "it took me a long time to realize that kids perceptions and experiences with the manips were usually so different from mine." She then raised issues about what adults and children bring to the use of manipulatives:

> I had a clear belief in the symbolic algorithms I had learned. I was delighted to be able to represent things I had always known in the real world. But kids are sometimes coming from the other direction. They are discovering relationships in the real world with manips, and then trying to learn to symbolically represent what they discover...

Four more responses followed, creating a string that moved back and forth between a second mathematics specialist, a mathematics educator, and this teacher. Each revisited issues that had been raised in this second message—the notion that children may make sense of representations in ways that can differ from how we see them, that manipulatives can serve to limit thinking if they are used too narrowly—but also introduced new and related issues. These included the observation that children may feel no discomfort with different solutions to the same problem:

> I have seen many children who think that an addition problem with base 10 blocks, such as 46 plus 25 is 71, but on paper it is 611, and see no problem with the fact that the two "answers" are different. Children have even told me: "This is the answer with the blocks and this is the answer with the pencil and paper."

and the observation that dialogue and discussion were of key importance as students moved back and forth between the "concrete/pictorial and the symbolic:"

What is interesting about this series of messages is the extent to which so many of these messages raised issues about how students learn. What is also striking is the extent to which messages—at least on the part of the teacher and the second mathematics specialist—involved examples of their own experiences using manipulatives to learn mathematics and observations of students working with these materials. Although the message from the mathematics educator did not contain these kinds of examples, her message did raise interesting questions about what was really happening when students seemed to be working at the abstract level:

> I'm intrigued. Is this where certain ways of thinking mathematically have been routinized, so it's not really problem-solving anymore? Or do we move from capturing a mathematical context with symbols, to operate on those symbols abstractly in order to find a solution, and then drop back down into the mathematical context again to make sense of the solution?

and it received a response that did include examples drawn from the teacher's classroom.

Message String #3: Ability Grouping

This message string began with a question posed by a teacher:

> My school ability grouped for math until two years ago. All of the literature we read seemed to discourage this practice so we shifted back to heterogeneous groups. I agree with this shift but now I am struggling with how to accommodate the many levels I have in my classroom...

In this message, she expressed concern about whether she was recreating, on a classroom level, the very same scenario they had been avoiding as a school. She discussed her feelings of discomfort with what she saw happening in her low group—"no sparks, no confidence, no ideas"—and ended by asking for help. This initial message set off a complex web of responses, many of these responses spawning new mini-strings within the message string itself. In these messages, issues were raised about (1) how students make sense of what other students say and do within mixed-ability groups; (2) the importance of student access to interesting and important mathematical discussions; (3) what we want students to be thinking about and learning as they work together; (4) the value of allowing for various perspectives; (4) ways in which the focus of small group discussions might be broadened through problems and tasks that get at important mathematical ideas; (5) helping students within the group learn to share their mathematical thinking; (6) the notion of involving students in assessing their own strengths and needs, and selecting their own groups; and (7) the importance of *students* thinking about what is important to learn.

Woven throughout the discussions were references to things teachers were doing or noticing in their own classrooms:·

> I really like the way this discussion is heading! It seems to me that the problems we present to children and the discussions we have about them are the key to handling diversity. Last year, when I was working with 2nd and 3rd graders, the math journals were wonderful vehicles for allowing children to discover that there are many ways to solve a problem .

Messages also included specific suggestions about the kinds of activities that might be productively explored within a small group, including a "good game that involves multiple operations and strategy development" and a "suggestion about multiple-step problem solving." Both of these—one from a teacher, one from a mathematics educator—were about activities that they had actually used with students. What is also interesting about this string of messages is that, as in the second string, there are references to teachers' own learning—in this case, their own learning as they worked on a mathematics problem in small groups during a TRN meeting:

> Nancy[3] "saw" the solution immediately... When it came time to share our results Nancy attempted to explain what she was seeing, but wasn't totally convinced herself ... A few others understood and were actually able to clarify the reason and bring even more of the group into the discussion. A sense of discovery was shared by many of the participants, even though Nancy was the initial discover.

This message, and the response to it, both raised issues about, and provided a way to think about, what it meant to be learning mathematics in a mixed ability group.

As in the earlier message strings, messages posted by teacher educators served to raise general issues that were not necessarily depicted through examples from actual classrooms. In this message string in particular, three mathematics educators created a mini-string of their own in which, in somewhat abstract terms, they discussing the importance of the kinds of tasks that are posed to students as they work in groups. While teachers did not participate in this discussion directly, the issue of the nature of the task did arise in subsequent messages in the chain, suggesting that teachers reflected on these points as well, and were later discussed them in the context of their practice. While a couple of the messages contained simply a "thank you for the suggestion," this message string consistently seemed to address a number of provocative and important questions about the issue of grouping students for mathematics instruction, as well as broader issues of teaching and learning mathematics.

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[3] "Nancy" is a pseudonym. No actual teachers' names are used in this paper.

Conclusion

There are several conclusions that might be drawn from the analysis of these three message strings. One is that the use of examples from actual practice—be it teaching practice with students or situations in which participants are learners themselves—had a way of grounding discussions in ways that lead to the consideration of different perspectives on issues. Secondly, a rich and productive message string was not necessarily begun by a detailed, provocative message—that it was possible to take brief beginning questions or comments and open them up in interesting ways. Thirdly, mathematics educators tended to participate in these discussions in ways that differed slightly from teachers and mathematics specialists, raising issues in a somewhat more abstract form. In some cases, these issues were ignored by participants, and in some cases these issues were placed in a context that allowed participants to think about them further.

However, there are still remaining questions about the use of electronic networks to support mathematics reform. How do we help teachers learn to create the kinds of electronic messages that allow for explorations of important issues in substantive ways? How are these cultures of inquiry created electronically, and in what ways are they connected to face-to-face relationships with colleagues? Finally, what is it that teachers gain from participating in these kinds of electronic networks, where dilemmas are framed and practice shared? These are just a few of the interesting questions that might be asked.

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THE IMPACT OF SCHOOL-STUDENT CASE STUDIES ON SECONDARY MATHEMATICS STUDENT TEACHERS' LESSON–PLANNING PRIORITIES

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An experiment was conducted to assess the impact on lesson-planning priorities of an activity designed to heighten student teachers' awareness of their school students. Twelve student teachers carried out the activity while 12 others served as a control group. The activity was a case study of a school student. An identical pre- and post-treatment questionnaire was completed by all 24 student teachers. Responses were analyzed to determine whether students' lesson-planning priorities had changed during a semester of student teaching and to compare the experimental and control groups. Results show no statistically significant differences in the lesson-planning priorities of the student teachers.

The Problem

While many tasks confront the teacher, Feiman-Nemser and Buchmann (1987) suggest that the central role of teaching is to promote learning. Despite years of personal experience as learners, they claim, it is difficult for prospective teachers to *focus on students*: "Perhaps most difficult is learning to shift from themselves as teachers or [from] the subjects they are teaching to what others need to learn" (p. 257). This difficulty is consistent with patterns of development suggested in the literature (c.f., Ball, 1988; Borko, Eisenhart, Brown, Underhill, Jones, & Agard, 1992; Brown & Borko, 1992), yet is fundamental to success as a mathematics teacher in a reform-based classroom. Teachers are expected to plan, execute, and reflect on lessons that build on students' existing knowledge and that incorporate assessment of student learning from multiple perspectives (National Council of Teachers of Mathematics, 1989, 1991, 1995).

This type of behavior does not occur naturally for most preservice teachers (c.f., Ball, 1988; Borko, Eisenhart, Brown, Underhill, Jones, & Agard, 1992). Brown and Borko cited the work of Feiman-Nemser and colleagues who urged that "teacher educators . . . take an active role in guiding pre-service teachers' pedagogical thinking and actions" (Brown & Borko, 1992, p. 217). The study reported on here assessed an activity intended to help preservice teachers gain "a concrete sense of pedagogical thinking and acting" (Feiman-Nemser & Buchmann, 1987, p. 257).

A Student Teaching Activity

The activity took place during a semester of student teaching at Illinois State University. It was designed to help student teachers recognize and value knowledge gleaned from school students and to use that knowledge in both one-on-one interactions and in whole-class settings. The activity required secondary mathematics student teachers to study, in depth and over an extended time period, one school student. The student teachers reported on the school-student case study and reflected on its impact on professional development.

Each case study required completion of five tasks: (1) confidential description of a school student; (2) analysis of the school student's mathematics work; (3) analysis of the school student's questioning patterns and the questioning patterns of the student teacher; (4) design, execution, and evaluation of a lesson, based on knowledge of the school student gained by the student teacher; and (5) reflection on the usefulness of the school-student case study.

Methods

An experiment was conducted to assess the impact of the school-student case studies on student teachers' lesson-planning priorities. The experiment was conducted with 24 volunteers from the 1995 Illinois State University secondary mathematics student teachers. The volunteers were randomly placed into one of two groups. An experimental group (n = 12) carried out the school-student case studies and the others formed a control group (n = 12) who completed a sequence of generic reflections.

To help determine whether student teachers' focus of attention had changed during student teaching, all 24 participants completed an identical pre- and post-treatment questionnaire that included this request:

1. List as many factors as you can think of that are important to consider as you plan and execute classroom mathematics lessons.
2. From your list, select three factors that are most important to you. Rank them 1, 2, 3, with 1 representing the most important factor. With each of the three factors, briefly state why it is important to you.

All responses (187 pre-treatment and 211 post-treatment) were sorted according to similarities and differences in their focus of attention. Based on several sort iterations and subsequent comparisons among the statements, six categories emerged to capture students' lesson-planning focus of attention. Descriptive labels and inclusion criteria were generated for each category (Day, 1995):

Categories for Students' Lesson-Planning Focus of Attention

Constraints (C): Lesson plan factors (LPFs) *beyond the control of the teacher*, in-cluding physical materials, elements of time and scheduling, and administrative decisions.

Content and Curriculum (CC): LPFs focused on *content* or on the *curriculum*.

Focus on Instruction (I): LPFs that identified *components of the teaching and learning process that occur within an instructional setting*, including strategies to be employed in the lesson, its organization, assessment, alternate approaches, and stumbling points.

Focus on Teacher (T): LPFs that identified some aspect of the *teacher's ability, personality, behavior, attitude, or another teacher characteristic*, including mathematics ability, management plan, experience, provision of effective transitions, and ability to listen to students' responses.

Focus on Students (S): LPFs with one or more students as the focus, identifying some *ability, personality, behavior, attitude, or other characteristic of the students*, including mathematics capability, familiarity with the learning setting, maturity, background, and the degree of homogeneity of a class.

Other (X): *LPFs that did not fit into any other category.* For example, "classroom environment" accompanied by no elaboration or description of its meaning and intent.

A second researcher was trained to categorize the statements according to these criteria. The author and the second researcher then independently coded each of the 144 statements identified in item 2 of the questionnaire for the pre- and post- treatment responses of the experimental and control groups. The researchers then shared the results of their independent coding. They discussed each of the statements and reached consensus on the categorization of 143 of them.

To test whether changes had occurred in lesson-planning priorities, data describing the number of statements in each of the six categories were analyzed using Fisher's Exact Probability Test (Siegel, 1956). This analysis technique was applied to the data to determine whether statistically significant changes had occurred in student teachers' lesson-planning priorities. Fisher's Test was used to test changes in responses from prior to to after student teaching as well as differences between the experimental and control groups.

Results

Changes in Lesson-Planning Focus from Prior To to After Student Teaching: The Entire Group of Student Teachers. Table 1 shows the distribution of pre- and post-treat-ment top-priority statements in each of the six lesson-planning focus categories identified in the analysis. The data in the 2-by-2 contingency table (Table 2) show the number of state-ments categorized as student focused (**in S**) and the number of statements not categorized as student focused (**not in S**) from both the pre- and post-treatment responses for the entire group (n = 24). Using Fisher's Exact Probability Test, the probability is 0.117 that these frequencies will occur, assuming no change in lesson-planning focus from prior to to after student teaching. Table 3 shows the distribution of the entire set of 144 categorized statements. Under Fisher's Test, the probability is 0.096 that these frequencies will occur, assuming no change in focus from prior to to after student teaching (see Table 4).

Table 1: No. of statements in each category (top priority)

	Category						
	C	CC	I	T	S	X	Total
Pre-Treatment							
Experimental	0	3	1	3	3	2	12
Control	0	4	3	1	3	1	12
Total	0	7	4	4	6	3	24
Post-Treatment							
Experimental	0	2	1	2	6	1	12
	1	1	1	3	4	2	12
	1	3	2	5	10	3	24

Table 2: Contingency table comparing stu-dent-focused vs non-student-focused state-ments (top priority)

	in S	not in S	
pre	6	18	24
post	10	14	24
	16	32	48
	p=0.117		

Table 4: Contingency table comparing stu-dent-focused vs non-student-focused state-ments (top three ranking)

	in S	not in S	
pre	18	54	72
post	23	49	72
	41	103	144
	p=0.096		

Table 3: No. of statements in each category (top three ranking)

	Category						
	C	CC	I	T	S	X	Total
Pre-Treatment							
Experimental	1	6	5	9	11	4	36
Control	4	8	7	7	7	3	36
Total	5	14	12	16	18	7	72
Post-Treatment							
Experimental	2	5	9	7	12	1	36
Control	3	8	5	5	11	4	36
Total	5	13	14	12	23	5	72

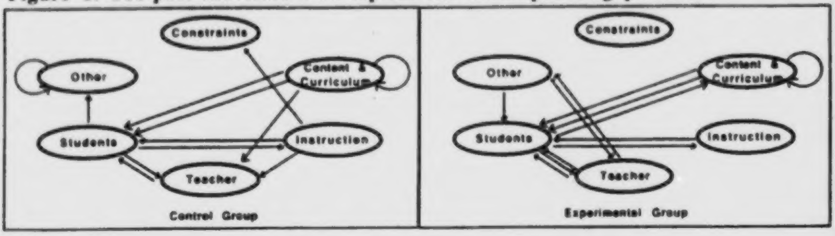
Comparison Between Experimental and Control Groups. Fisher's Test was also used to compare the control and experimental groups. Again using the frequencies shown in Tables 1 and 3, 2-by-2 contingency tables were subjected to Fisher's Test. The probabilities shown in Table 5 suggest no differences in lesson-planning focus between the control and experimental groups either before or after the treatment.

Table 5: Group comparison over time: student-focused vs non-student-focused lesson-planning priority statements

Time	**probability of given frequency**
Pre-treatment	
top-priority statement	0.360
top three statements	0.121
Post-treatment	
top-priority statement	0.233
top three statements	0.193

Pre-Treatment to Post-Treatment Movements. Movements among the lesson-planning focus categories were analyzed next. Figure 1 is a pictorial representation of the change in top-choice lesson-planning priority for members of the control and experimental groups. Each directed segment represents the change in focus for one student teacher based on pre- and post-treatment questionnaire responses. Observation of the movements to and from the student-focus category led to statistical analysis of that movement.

Figure 1: Pre-post movements in top-ranked lesson-planning priorities



Fisher's Test was used to compare the movements of the control and experimental groups. A 2-by-2 contingency was generated from the information in Figure 1 to show the number of control and experimental group members who *did* change the top-choice lesson-planning priority (10 and 11, respectively) as well as the number of those who *did not* (2 and 1, respectively). The probability is 0.391 that these frequencies will occur, assuming no difference in the movements of the group members.

Table 6: Experimental vs control group comparison of movements in top-ranked lesson-planning priorities

Comparison	probability of given frequency
Top-ranked priority: changed or not	0.391
Movement into S vs movement into any other category	0.275
Movement out of S vs movement out of any other category	0.365

For those student teachers whose top-choice lesson-planning priority *did change* from pre- to post-treatment, Fisher's Test was used to compare pre-post movement *into* the student-focus category and movement *out of* the student-focus category by control and experimental group members. Probabilities shown in Table 6 indicate no differences in change of lesson-planning focus between the control and experimental groups.

Conclusions and Discussion

Based on the categorization and subsequent statistical analysis of the student teachers' pre- and post-treatment statements of lesson-planning priorities, the school-student case studies conducted by 12 of the 24 student teachers had no impact on the teachers' focus of attention on their students. In all comparisons, those of the pre-post responses of all 24 student teachers and those of the control-experimental groups, Fisher's Test showed that the resulting frequencies were not unlikely for groups that shared no differences.

Several possibilities exist to account for this. Perhaps the all-encompassing nature of the student teaching experiences (Maxie, 1989) had more significant impact on the preservice teachers than whether a student teacher took special focus on a school student throughout student teaching. The case-study tasks assigned to the experimental group members were intentionally designed to require little extra involvement by a university supervisor or a cooperating school teacher; it could be that more active involvement would make the case study tasks more meaningful. It also must be considered that the tasks were simply ineffective and of little value to the student teachers.

In addition to designing research to explore these possibilities, further analysis of both the experimental-group case study task responses and the control-group generic reflections may reveal factors that have significantly impacted upon the preservice teachers during their student teaching experience. The results of the present study leave no doubt that generating activities that sustain long-term impact on student teachers is no small task, yet one that continues to require the attention of those involved in mathematics preservice teacher education (National Research Council, 1996).

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CONSTRAINTS ON CONSISTENCY BETWEEN EPISTEMOLOGY AND PRACTICE

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This study was conducted in order to gain insight into an educator's framework for interpreting her instructional experiences. Findings indicated that student response was among both the greatest supports for and constraints on the participating teacher's instructional practices. Further, her ability to justify her decision to de-emphasize interstudent discourse and nonroutine problem solving suggests that constructivist epistemologies may need to incorporate a process for distinguishing between competing knowledge claims in order to better facilitate teachers engaging in practices that reflect the use of such epistemologies as a referent for their pedagogies.

This paper is based on a case study of Sharon, a middle school mathematics teacher who endorsed a constructivist epistemology, was "not book driven" and was striving to implement nontraditional instructional practices, i.e., methods consistent with those recommended in the National Council of Teachers of Mathematics (NCTM) Curriculum and Evaluation Standards (1989). The purpose of the study was to gain insight into Sharon's framework for interpreting her instructional experiences by investigating both her constructions of experiences that she described as sustaining or constraining to her efforts to teach nontraditionally and her justifications for these constructions. Implications were used as the basis for making some recommendations for assisting teachers in empowering themselves as critical reflectors and thereby, enhancing their potential to detect and eliminate discrepancies between their instructional beliefs and practices.

Methodology & Interpretive Framework

Since, in teaching, an educator learns about her students' knowledge and learning styles, and since an educators' efforts to alter her beliefs and practice entail learning or cognitive reorganizations of her model of experiential reality and the actions that model informs, my interpretive framework was largely based on the premise that learning is inherent in teaching. Accordingly, I chose to utilize radical constructivism and Shaw & Jakubowski's (1991) model of teacher change as major components of that framework. Data were primarily collected via participatory observations and semi-formal interviews, with interview questions focusing on Sharon's instructional practices as well as her beliefs concerning teaching and learning. Later, I reviewed interview transcripts in search of statement patterns that could serve as the basis of a grounded theory both of how she interpreted her instructional experiences and of factors that influenced those interpretations. Finally, I presented my theories to Sharon in an effort to obtain respondent validation.

Student Response and Resistance

Sharon's students' responses to having participated in her class were among the experiences that she found most sustaining to her efforts to teach nontraditionally. She found it "very rewarding" when learners found her class "interesting, fun and challenging" and felt like they had "learned something." Moreover, she asserted that "if students aren't generally happy with their situation, then they won't be able to think through a task or learn something new or work on something with the concentration they need" to be successful. The rewarding feeling that Sharon drew from her pupils' generally positive responses to having participated in her class, in conjunction with the relationships she perceived between learners' happiness, motivation and success encouraged her to continue to strive to utilize activity-based instruction and thereby, supported her efforts to teach nontraditionally.

Even so, one of the most powerful constraints on her efforts to implement nontraditional teaching methods arose out of an interaction between the relationship Sharon perceived between learners' happiness, motivation and success and her students' resistance to becoming engaged in interstudent discourse and investigations of nonroutine problems. Certainly, this is a constraint with which most, if not all, educators wrestle. However, in making assertions such as "They won't contribute, they won't try, and they don't care," and "It's not that I don't think they're capable of it. It's just that they seem to choose not to use their capability," Sharon focused on what her learners would not do and ignored what she could do to improve the situation. Accordingly, this indicated that she constructed student resistance as an external constraint i.e., a constraint that she was powerless to alter or remove. Consequently, Sharon could not overcome this obstacle, and it is unlikely that she will do so until she reconstructs student resistance as a constraint that is not completely external. Likewise, it is improbable that she will overcome student resistance until she personalizes or gains ownership of her conception of the ideal learning environment by constructing a vision of an alternative reality for her classes and projecting herself and her existing instructional context into that vision (Greene, 1988; Shaw & Jakubowski, 1991).

Sharon's construction of student resistance as an external constraint coalesced with her desire to create an atmosphere that was permeated by the beliefs that "everyone can succeed and everyone can learn" and the relationships she perceived between learners' happiness, success and self-confidence to form the foundation on which she constructed her justification for her decision to de-emphasize interstudent discourse and nonroutine problem solving. Moreover, she operated under the premise that her students had very little confidence in their abilities to do mathematics and therefore, needed experiences in which they could be successful to bolster their confidence; for without confidence, they would not make genuine efforts to resolve challenging problems and would not be "willing to risk making mistakes." Accordingly, Sharon perceived that student resistance was indicative of unhappiness which led to a lack of success, lower self-esteem and increased resistance, thus constituting a vicious circle. Hence, it seemed reasonable to Sharon to only infrequently attempt to engage her learners in such activities.

Moreover, the student resistance Sharon encountered when she endeavored to engage her pupils in interstudent discourse or nonroutine problem solving was not displayed when she modeled or explained problems, processes or concepts. This in conjunction with the value she placed on modeling as a method for facilitating learning, and thereby, for nurturing her learners' sense of success and confidence in their mathematical capabilities, provided her with a moral as well as logical rationale for de-emphasizing problem solving and discourse. Hence, Sharon tended to model solutions to problematic situations and provide explanations of the concepts and processes involved; rather than to strive to engage her learners in collaborative investigations of such problems and in discussions of their mathematical ideas. Of course, modeling can facilitate learning, and it is perfectly acceptable to model solutions in an effort to bolster students' confidence and help them learn to resolve problems. However, routine and modeled nonroutine problems need to be supplemented with challenging situations in which the learners are expected to take responsibility for their learning, and it was in this regard that Sharon was inconsistent with the recommendations of the *NCTM Standards* (1989) and her conceptions of both ideal instructional practices and the ideal learning environment.

Epistemological Considerations

The discrepancy between Sharon's instructional practices and espoused beliefs, that arose out of her decision to infrequently attempt to engage her learners with potentially problematic situations and interstudent discourse, is not an isolated occurrence (Thompson, 1984). Moreover, her making this decision despite her endorsement of constructivism, a theory which implies that such activities are likely to facilitate learning, raises the question that if knowledge of radical constructivism is not sufficient for teachers to implement instructional practices that are consistent with that theory, what can teacher educators do to empower educators to be able to utilize practices that are consistent with the theory.

Various researchers (Russell & Corwin, 1993; Simon & Schifter, 1991; Tobin, 1990) have sought to answer this question by striving to help teachers to critically reflect on their pedagogical beliefs and metaphors as well as the implications those constructs have for the classroom roles of teachers and learners. Moreover, many of these researchers were able to facilitate a change in teachers' beliefs and practices (Russell & Corwin, 1993; Simon & Schifter, 1991; Tobin, 1990). While this method may facilitate teachers becoming critical reflectors, a question that now arises is how else can teacher educators help teachers empower themselves as critical reflectors and thus, enhance the likelihood that teachers will be able to autonomously detect and eliminate discrepancies between their educational beliefs and practices? Recall that Sharon's decision to de-emphasize nonroutine problem solving and interstudent discourse was the product of interactions of multifarious factors, and that it is improbable that she will realize her vision of the ideal learning environment until she reconstructs student resistance as a constraint that is not entirely external. A similar assertion holds for her justification of her decision to infrequently seek to engage her learners in collaborative investigation of nonroutine problems and interstudent discourse of their mathematical ideas. In light of Sharon's actions and needs, one must question whether

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knowledge of radical constructivism, or any nonobjectivist epistemology, is sufficient to empower one to deconstruct one's beliefs and their associated justifications, or the subtle, yet powerful, myths underlying objectivist epistemology that impede the use of the theory as a pedagogical referent. Moreover, one must question whether such knowledge can provide one with sufficient impetus to alter one's pedagogy to reflect the use of constructivism as a referent for those beliefs and practices.

Certainly, radical constructivism seeks to explain how people learn and therefore, has implications concerning how to facilitate the deconstruction of beliefs and cultural myths. Nevertheless, just as it does not prescribe methods for teaching, it does not prescribe a method by which one might facilitate such deconstructions. Moreover, radical constructivism does not endeavor to provide a method for making qualitative evaluations of viable constructions. Constructions are either viable or they are not; therefore, if a qualitative or value judgment is to be made between competing constructs, it must be made on the basis of a criterion of assessment other than viability. Accordingly, radical constructivism does not offer a means for differentiating between the worth of competing knowledge claims which limits its utility as a referent for making such judgments, and thus, its utility as a referent for altering one's instructional practices. The nearest radical constructivism comes to offering a means of judging the value of competing knowledge claims is via its assertion that one's constructs must meet the socio-physical constraints of one's environment. However, since language can be used to persuade or manipulate others for self-serving ends, and since language mediates the negotiative process through which the collective creates the social constraints of knowledge, the individual is largely at the mercy of the group when it comes to assessing the worth of competing viable constructions or claims concerning the nature of knowledge and learning (Hardy & Taylor, 1996).

While radical constructivism does not offer a means for making value judgments between competing knowledge claims, Habermas' theory of communicative action does provide such a method. Accordingly, it is possible that radical constructivism could offer a general process for distinguishing between competing knowledge claims if it incorporated the concept of engaging in communicative discourse for the purpose of arriving at mutual understanding that leads to cooperative actions that meet both group and individual needs, while protecting all parties from self-serving competitive interests (Hardy & Taylor, 1996). However, extreme caution must be exercised as the theories are not entirely compatible and therefore, cannot simply be appended to each other.

One point of divergence between radical constructivism and Habermas' theories lies in his constructs of knowledge-constitutive-interests, and since radical constructivism is inherently concerned with knowledge construction, it is important to consider such a discrepancy. A fundamental assertion in Habermas' theory of knowledge-constitutive-interests is that knowledge is interest specific (Ewert, 1991). That is, the requirements that a construct must meet in order to be characterized as knowledge are dependent upon the interest one utilizes for interpreting and organizing one's experiences. Further, each interest utilizes a distinct rationality or scientific method to resolve questions concerning the validity of knowledge (Ewert, 1991; Habermas, 1984). Accordingly, Habermas' theory of knowledge-constitutive-interests focuses on how people *interpret* mental constructs and use those interpretations to organize their experiential flow, rather than on the general process by which people *synthesize* mental constructs and use those constructs to interpret and organize their experiential flow, which is the focus of radical constructivism. Hence, the theories do not address identical issues, so care must be taken in trying to relate them.

Although the above inconsistency is associated with Habermas' theory of knowledge-constitutive-interests rather than his theory of communicative action, another, more crucial, difference between Habermas' theories and radical constructivism does concern this theory. In short, communicative action is individuals' use of discourse to arrive at a mutual understanding of one another's assertions as well as the implications of those assertions and the assumptions on which they are based, in an effort to coordinate action that will meet the needs of both the individual and the group (Habermas, 1984). Further, communicative rationality is the process by which "validity claims about the rightness, truthfulness, or sincerity of a speech act,...can be raised and responded to in discourse," (Ewert, 1991, p. 359). Moreover, the communicative rationality associated with an interest can be utilized in conjunction with the requirements that the interest's science imposes on constructs purported to be knowledge to resolve such questions of validity.

It is the issue of validity that is problematic from the radical constructivist perspective. Recall that any construct that meets the socio-physical constraints of the cognizer's environment and thereby, fulfills its intended purpose is viable. Therefore, if multiple constructs allow one to attain the same intended goal-state, they are all viable. Further, since the only criterion for assessing constructs' viability is their fulfillment of their intended purpose, it is meaningless to speak of one viable construct as being more or less viable than another. Thus, viability cannot be used to make qualitative distinctions between constructs which fulfill their intended purpose (von Glasersfeld, 1984). Moreover, a basal premise of radical constructivism is that people have no inherent ability for accessing the structure of objective reality and hence, cannot verify a construct's or knowledge claim's match with that structure. Consequently, questions of ultimate truth or correctness cannot be answered and are therefore, moot issues. Accordingly, the theory of communicative rationality cannot be appended to radical constructivism, nor can it be integrated with the theory without first modifying either the concept of viability, which would destroy the integrity of radical constructivism, or the concept of validity, by removing its associated concepts of ultimate truth or correctness.

However, the situation may not be hopeless, since under Habermas' theory, that which constitutes validation is determined by the requirements of the rationality used to inform the science of the interest in which one operates. Therefore, it is reasonable to conclude that there is room for variation in the definition of validity under the Habermassian theory. Consequently, there remains some potential for partially integrating aspects of or modified versions of the theories. The incorporation of some form of communicative rationality with some form of constructivism could provide the latter with a process through which people can empower themselves to judge between competing knowledge claims, deconstruct personal beliefs or cultural myths and perhaps, create compelling moral and logical justifications for implementing nontraditional strategies that reflect the use of some form of constructivism as a referent for their pedagogies. Nevertheless, such endeavors must proceed with caution. Further, if components of a constructivist theory are to be utilized, the concept of viability must be retained; however, a compatible conception for validity or moral viability will be needed for the endeavor to have any chance of success.

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THE DECATUR ELEMENTARY MATH PROJECT

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The Decatur Elementary math Project (DEMP) was designed to prepare two lead mathematics teachers in each of the seven elementary schools in City Schools of Decatur, Georgia. Using the teacher education model developed for middle school teachers in the Atlanta math Project, which is based on the assumptions of constructivism, metacognition and shared authority, the two project directors of DEMP engaged fourteen teachers in a one week staff development, in monthly academic year workshops, and in monthly academic year plan/teach/debrief sessions in the participating teachers' classrooms. This paper describes how using the lens of sharing authority provides one avenue for interpreting change in a teacher's classroom. Using this lens I will describe how Jewell, a kindergarten teacher, changed her practice in each of the domains of the NCTM teaching standards.

Background

In response to reform in mathematics education, the teacher change process has become the focus of many studies in recent years. It is clear from this work that the kind of change envisioned in reform documents involves more from teachers than learning a new set of teaching strategies and more from teacher educators than simply lecturing teachers about what they ought to do. Teachers are being asked to change what they believe it means to know mathematics and what it means to do mathematics. They are being asked to change the culture of the classroom. Teacher educators are needing to model a different kind of classroom experience. This kind of fundamental change does not come easily.

The Atlanta Math Project (AMP), National Science Foundation, 1990-1995, (Hart & Schultz, 1989) developed a teacher change model grounded in the assumptions of constructivism as a theory of how we learn, metacognition as a theory of how we think, and of sharing authority as central to the success of the teacher change process. In AMP these assumptions frame all the experiences and the interactions of the participants. Teachers must have experiences which allow them to construct or reconstruct their beliefs about teaching and learning mathematics--just a students must construct their own mathematical understandings. Change is enhanced if nurtured in a reflective environment where teachers consider their practice and understandings and monitor and regulate that practice during the teaching act. And, the ability to share the intellectual power of the classroom is critical both in the experiences teacher educators provide teachers and the experiences teachers provide students.

Using this model I worked with City Schools of Decatur, a small urban school system within the boundaries of Atlanta, to develop a teacher enhancement project for their elementary school teachers. Fourteen teachers were funded by the State of Georgia's Department of Education to participate in the Decatur Elementary Math Project (DEMP) during the 94-95 school year. Two teachers, one each from the lower division (K-2) and upper division (3-5), from each of the system's seven elementary schools volunteered to participate. The goal of the project was to develop lead mathematics teachers in each school who could support the vision of the reform. I co-directed the project with the mathematics coordinator of the system.

DEMP began in the summer of 1994 with a one-week staff development that provided a range of experiences for the teachers. The teachers studied NCTM teaching standards (National Council of Teachers of Mathematics, 1991). They observed planning/teaching/debriefing sequences using classrooms from summer school mathematics classes. They engaged in paired problem solving sessions where they learned to listen to other teachers strategies and approaches to problems and to share their own strategies and approaches. And, they participated in heuristic teaching experiences. While all of this provided a useful grounding for the teachers, the classic work of Joyce and Showers (1908) and work in earlier projects (Atlanta Math Project, 1989-1995; Metacognition, Teachers and Problem Solving, 1985-1989) taught us that little real change in classroom practice occurs from one-shot staff development. At the end of the week the teachers were able to talk and think about the experience, but few were able to translate that experience into classroom practice.

As follow-up during the school year the teachers attended monthly staff development sessions which focused on the mathematics of one of the NCTM K-4 Curriculum Standards. The teachers also developed teaching partnerships with me or the mathematics coordinator by participating in monthly planning/teaching/debriefing sessions. Each month we planned a lesson together; the lesson was taught with the non-teaching partner observing; and we debriefed on each lesson. Initially I taught the lesson, offering the teacher a chance to watch the outcome of our collective thinking. Later in the year the teacher taught the lessons and received support from me. The plan/teach/debrief process was designed to be non-evaluative and to offer the teacher the opportunity to think about what worked and what did not work in her lesson, to think about her students' understandings, and to consider what things she might do differently in future lessons. Each lesson was video taped and teachers had the opportunity to review their lessons. Many times videotape was used as part of the monthly staff development session.

Sharing Authority

As mentioned earlier, the teacher education model that was developed in AMP and used DEMP was based on the assumption that sharing authority is a critical factor in creating classroom environments that mirror the image of reform. The domains of teacher change--whether they be change in mathematical knowledge, change in pedagogical knowledge, or change in practice--are all important to study for the professional community. However, the underlying factor that *seems* to impact the change in all of these areas is the ability of the teacher of the teacher educator to share the intellectual authority of the classroom--to encourage and value various perspectives, to honor knowledge gained through experience, and/or to respect the reasoning and thinking of others. Since this perspective framed all of the teachers' experiences in DEMP, I expected over time to see evidence of this outlook in the teachers' classrooms.

In an earlier paper (Hart, 1993) I described using shared authority as a framework for analyzing change in teacher practice. In that paper I describe how classroom dialogue in whole class discussion is different as a teacher begins to open up and respect, honor and encourage varying ideas. For the purpose of this paper, I will use a more holistic perspective and describe the results of the analysis of two videotapes from on teacher n DEMP. Using the perspective of sharing authority I will attempt to describe how classroom practice has changed with respect to the first five teaching standards.

Each teacher in DEMP was asked to make a video tape of themselves teaching a "good" mathematics lesson in the spring before the project began. At the end of the first year of the project the teachers submitted a second self-made video tape of themselves teaching a "good" mathematics lesson. The rationale for letting the teachers make their own tapes was to get a glimpse of what they valued and what they considered *good* before and after participation in the project. Although data was collected from all fourteen teacher, will be used to illustrate the process of analysis and describe the evidence of change in one teacher.

A Classroom Example

Jewell is a kindergarten teacher with 20 years experience inthe classroom. She volunteered for the DEMP project and proved to be a willing participant. During inservice sessions she was always eager to share her thinking and to listen to the ideas of others. Her nature was very non-authoritarian. The two tapes Jewell made of her teaching capture subtle differences in her classroom before and after the project.

The classroom environment. In the tape of her first lesson Jewell had seven students seated closely around a rectangular table with Jewell seated at the end giving instructions. The students were asked to perform tasks and the teacher reviewed their work by glancing around the table. The accepted pattern of behavior was fro Jewell to ask questions, for the students to respond chorally, and for Jewell to acknowledge the correct answer. If the correct answer was not received, Jewell would then give it herself. For example, one child had three seeds on her card and Jewell asked the group, "How many more do you need to make five?" Several students responded, but Jewell passed over their remarks and quickly answered her own question responding, "You'd need two more, wouldn't you."

In the second tape Jewel shows us a whole-class lesson with students sitting on the rug around the floor. She demonstrates the activity they will engage in, passes out materials and then the children are sent in pairs to work independently on their investigation. Upon completion the partners return to the group and share their results. Jewell provides time for each group to work and for each group to explain their results.

In both lessons Jewell presents very clear instructions about the activity she wants the children to perform. In the latter tape the children have more freedom to design their own strategy and to share that strategy with the class.

Tools for enhancing discourse. In both tapes Jewel uses manipulatives with the children. In her initial tape Jewell is seated around a table with students doing activities with seeds and pictures of watermelons. The students are to place the correct number of seeds on the watermelon as she displays various numbers. The cards and seeds provide a visual to help the students point and count out loud to determine their answers.

In the second tape Jewell is engaging the students in an "investigation." The students are to toss 20 coins and records the results on a bar graph or on chart paper using tally marks. They work in pairs with one student recording and one student tossing.

In both situations the materials helped produce dialogue about the mathematics, however, there were subtle differences in *how* they were used. In the first tape the students were directed to put the 3 seeds they stared with on one side and the 2 additional seeds they needed to make 5 in a separate pile. The instructions were fairly directive, "Why don't you put the seeds like this..." The students had little freedom to choose their arrangements or strategies. In the second tape the students were free to choose their own method for using the materials. They could choose who tossed and who recorded.

The teacher's and students' role in discourse. Taken out of context Jewell's questions in the initial tape sound very open-ended. She asked questions like: *What did you do to find out? Tell me what you did to get 8 when you had 5*? However, a short snapshot of the dialogue demonstrates a very closed environment where the children have no opportunity to share their thinking.

Jewel begins by holding up the number "3" and instructs the children to place this many seeds on their cards. She visually checks the cards and then has the children verify their own by counting and touching them. She then holds five and asks them to show this number on their card. She asks, "How could you do it?" Several children want to remove the 3 seeds and start over, but Jewell is anxious for them to use the counting on strategy and dismisses that strategy very quickly with, "Well you could but if you left the three on how many more would you need?" Several children respond with "5". Jewell passes over their response and asks the question again. When no one gives the correct response she says, "You'd need two more, wouldn't you."

In the second tape Jewell not only allows students to explain their results for the investigation, she also takes advantage of some incidental learning and permits a boy to explain his thinking on a problem he has solved. Jewell had planned to toss 20 coins but by mistake when she counted up her heads and tails she got 12 and 9. She exclaims.

> Jewell: "That tells me I might have made a mistake about how many pennies I had in the bag. Do you think 12 and 9 make 20?"

Kim: "That's 21."

Jewell: "How do you know that Kim?"

Kim: "Just take 9 and 12. Take 2 off of it--the twelve. Now you have 10, 9 and 2. Take one penny off the two and stick it on the 9 and that makes 10 and 10 and 1 and that makes 21."

At no time during the first lesson did a child share his or her thinking like Kim did in the second tape. Jewell was much more encouraging of students' sharing their ideas and strategies.

Summary

In these brief examples I show how at the end of the project year the culture of the classroom has changed, how manipulatives are used in a more open way and how students are encouraged to share their thinking and participate in the discourse. Jewell is beginning to share the intellectual authority of her classroom and the result is an environment that more closely resembles the image of reform and the standards.

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MATHEMATICS PEDAGOGY IN GUATEMALA: THE NARRATIVES OF TWO INNER-CITY TEACHERS

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The goal of this study was to shed light on the complicated and dynamic issues that affect the pedagogy of secondary-level mathematics teachers in Guatemala, Central America. Two inner-city teachers shared stories with me over the course of two summers about their working conditions, and their pedagogical goals and practices. Themes emerged from the teachers' narratives revealing the factors which impacted their work, including: the government's lack of support for education, the teachers' need to supplement their incomes, their inability to collaborate, and their inadequate access to research in mathematics education. The teachers' conventional pedagogy reflected the context in which they labored. A research agenda is outlined to continue the process of studying how Guatemalan mathematics teachers can be supported.

Introduction

This research report is based on interviews and observations conducted during the summers of 1994 and 1995 with two secondary mathematics teacher who worked in public, secondary schools in Guatemala City, Central America. The teachers, Señora Rodriguez and Señor Chavez, both had more than 20 years of experience teaching mathematics. This study is intended to begin a chain of inquiry, with the specific goal of commencing to answer the following question: What sense do two Guatemalan mathematics teachers make of aspects of their work and the contexts in which they work? More specifically, what narratives concerning their professional work contexts, goals, and pedagogical practices do the teachers offer as part of their sense making efforts?

Altbach, Arnove, and Kelly (1992), Carnoy (1974), and Zachariah and Silva (1980) have all argued that national school systems are situated within the context of unequal power relations among nations. The historical roots of this unequal relationship date back to when colonialized countries adopted transplanted educational models that usually "grotesquely misfit the countries' actual needs, circumstances, and resources" (Coombs, 1985, p. 33).

The need for a Third World perspective on mathematics pedagogy is based, primarily, on the lack of participation of developing nations in deciding the educational models that are most appropriate for their needs. Constructing educational systems that are responsive to local needs entails people defining "their own issues in their own ways, from their own perspectives, using their own terms" (Secada, 1995). My hope is that the teachers' insights help inform work undertaken to support mathematics education in Guatemala.

Introduction to the Schools and the Research Participants

The "Institute" was a public, all-girls, secondary institution in Guatemala City. The Institute awarded the "magisterio" diploma, a degree certifying primary-school teachers. The "Gymnasium" was a public, all-boys, secondary institution also located in Guatemala's capital city. It was one of only a handful of public schools in Guatemala that awarded the "bachillerato" diploma, the most advanced, pre-tertiary degree offered in the country.

Sra. Rodriguez had been at the Institute for only three years when this study was undertaken, but had extensive experience as both a mathematics and science teacher. She was a very amiable person who interacted easily with staff, students, and friends. Sra. Rodriguez was very polite and always aware of social mores, but also possessed an excellent sense of humor and enjoyed laughing.

As the senior member of the Gymnasium faculty, Sr. Chavez was the most important leader of the staff. Sr. Chavez had a strong presence that differentiated him from others. Given Sr. Chavez' genuinely warm personality and obvious integrity, it was understandable that both faculty and students depended on him for guidance.

Overview of the Research Paradigm and Methodology

In the spirit of the critical research tradition, this work was carried out in "partisanship in the struggle for a better world" (Kincheloe & McLaren, 1994, p. 140). The approach to voice scholarship chosen was narrative inquiry. In narrative inquiry, how the teachers frame their answers provides the structure to describe their words (Riessman, 1993). My review of the literature revealed that no research exists in which mathematics teachers from developing nations depict their work in their own words.

Given the level of distrust among faculty at both the Institute and the Gymnasium, direct observations of Sra. Rodriguez and Sr. Chavez in the classroom were limited. I observed Sra. Rodriguez teach on five separate occasions in the summer of 1995, and attended four of Sr. Chavez' classes during the summer of 1994. Structured and unstructured, open-ended interviews with the mathematics teachers were conducted entirely in Spanish. These afforded the primary data for the study.

An issue that became salient during the data analysis was the difficulty that I encountered in applying a critical theoretical perspective to the sense that the teachers made of their work. The themes that emerged in the teachers' narratives, not a critical theoretical framework provided the basis from which I made sense of the teachers' narratives.

The Study's Findings

According to Sra. Rodriguez, her primary goal was to prepare her students, all of whom were female, to learn "the basics" of mathematics. Since her students were all working toward a primary school credential, she believed that her students needed to understand the basics to be able to teach children the math facts. Regardless of her students' grade level, Sra. Rodriguez began each class by assuming that her students knew little mathematics and spent considerable time requiring her students to practice their math facts. She perceived herself as willing to work with all incoming students despite their academic weaknesses, but she also expected them to adapt and learn at the instructional pace that she established in class:

> The premise when I begin teaching is to assume that the student knows nothing. I draw a big zero [on the board] and I tell them that in my class nobody knows anything and anyone of the young women who are more advanced can go to a higher level class and that the ones that are below the class level can go to a lower level class. So, I start from scratch assuming that no one knows a thing, then I'll start with the simplest topic so they can understand from the beginning until things start to get more difficult. Then if they lose the flow of the teaching and they make math difficult, then that's their problem. This is no longer the instructor's responsibility.

During all of the classes which I observed, Sra. Rodriguez initiated class by demonstrating how to perform specific calculations or manipulate equations and proceeded by having her students complete an example. She claimed that she followed the government-mandated mathematics curriculum closely. Since she had participated on committees to determine the national mathematics curriculum, Sra. Rodriguez has had opportunities to emphasize the importance of basic skills instruction to her colleagues. Sra. Rodriguez also did not teach with the aid of calculators. The only jobs that her students found were in rural regions where calculators were unavailable. She could imagine though, that if her country was "technologically advanced like Japan or the United States" that she would be more likely to utilize technology to teach mathematics.

Another important goal of Sra. Rodriguez' was "to prepare them [her students] to be well-prepared, find a place in society and if this doesn't happen it's because they don't study to distinguish themselves for anyone. In some cases, it's not possible for some girls. For the great majority it is and they become competent professionals." Sra. Rodriguez had succeeded in a male-dominated discipline. She was a success and her students could also thrive if they devoted themselves to their studies:

> Success . . . to me means - I repeat myself again - is to find a former student out there in the real world as a professional. That to me is a grand success because these former students have within their professional lives something I taught them. Downfalls... I feel like I've never experienced any because I have never been accused of being a bad teacher. They do complain about their final grades sometimes, but never the quality of my teaching. I was especially successful with my male students because in other teachers' classes a lot of students will miss classes but in mine they would all show up. Having a full class is a success. That they don't all make it through the class is a different thing, because not all of them have the same degree of dedication. So about downfalls, I cannot speak of any. On the contrary, I feel like a successful woman because I have done all of this on my own - a very successful person.

Sra. Rodriguez indicated that her practices were minimally affected by the notorious political activities of her students. An interesting consequence of the strikes and the government's occasional tardiness at paying the teachers' salaries was that teachers developed other professional skills. Sra. Rodriguez worked as a seamstress in the

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afternoons. Because teachers worked other jobs, they had little time to work collaboratively.

According to Sr. Chavez, his primary goal was to mathematically prepare his students, all of whom were male, for success in the workplace and at the university. Sr. Chavez worked hard to develop what he characterized as his students' "reasoning and deduction capabilities." The study of mathematics fostered students' reasoning skills, and in Sr. Chavez' opinion, could lead to the realization of changes in other areas of people's lives. Sr. Chavez illustrated his faith in the power of the study of mathematics in the following:

> The majority of people in a conversation use expressions such as this one, "all of the professors at the Institute are drunks." Why? (in response to the individual's declaration) "Because, I saw one at the bar last night." This is the form that the majority of the population expresses their opinions. After commencing to study mathematics, the responsibility that exists when using a universal quantifier can't come from only a few cases. When they understand this, their logical abilities increase tremendously.

Sr. Chavez believed that in Guatemala, children did not begin to think abstractly until between 14-16 years of age, late in Piaget's developmental scheme. The principal reason for this was that Guatemalans had been poorly fed for generations resulting in biological deficiencies. One of Sr. Chavez' goals was to stimulate his students to begin to think more critically through the study of algebraic abstractions and generalizations.

Sr. Chavez told me that he taught at the Gymnasium because of his concern for his students' many needs. He identified very strongly with his students, illustrated by his comment that "I'm one of them now." Not only did he speak the street language that his students spoke, he also served as a friend to his students:

> To begin with, when I enter the classroom the first day, I don't present myself as a teacher. I present myself as a friend who is there to help them. And the second thing that they understand is that I'm there everyday to teach them. Actually, for them this is a surprise. [Also], I must learn their language [idiom]. I must learn it - they have a language that's very particular. If a person goes to their classroom and speaks formal Spanish, they can't expect to achieve anything. For example, for bus they say `burra,' bike - `biula,' police - `tirra.' You must understand these words of the street [to effectively teach these students].

In contrast to Sr. Chavez, the government showed little commitment to public education. Sr. Chavez had never been visited by personnel from Guatemala's Ministry of Education. According to Sr. Chavez, because of the lack of supervision by the state, mathematics curricula varied greatly among schools. "In Guatemala, it's totally disorganized. If you went to one school in one place, they change from year to year. In other schools [it's] the same, everyone teaches how they want, there's no order." Sr. Chavez depended on a textbook that was written more than 30 years ago that functioned as his de facto curriculum.

Final Remark and Next Steps

The teachers' traditional pedagogical practices reflected the context in which they worked (e.g., need to work additional jobs, lack of supervision and classroom support, no teacher collaboration). Nevertheless, Sra. Rodriguez and Sr. Chavez were remarkably effective mathematics teachers. Their powerful stories, narratives from their students which demonstrated their satisfaction with Sra. Rodriguez' and Sr. Chavez' pedagogy, and my observations in the teachers' classrooms confirmed this finding. Given the harsh conditions in which they labored, their conventional means to teach mathematics should not be used to diminish their triumphs.

Both teachers indicated that they would like scholars from the United States to coordinate workshops with Guatemalan mathematics educators on the teaching and learning of mathematics. Alternative means to teach mathematics and assess students' mathematical knowledge should gradually be explored in these in-services. Efforts should also be initiated to ensure that teachers have some means of occasionally communicating with and supporting one another. Research needs to be done to gauge the effectiveness of these programs.

The fact that as many as 30% of the students at the Gymnasium (by Sr. Chavez' estimation) were from rural, indigenous backgrounds provides some inspiration to find the means to advance programs that could impact the retention rates of these students. In general, continuing the process of learning about mathematics pedagogy in Guatemala is crucial to support the work of Sra. Rodriguez, Sr. Chavez, and their colleagues.

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ALTERNATIVE ROLES FOR PARTICIPANTS IN FIELD EXPERIENCES

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This paper presents the results of a research study involving preservice elementary teachers in a non-student teaching field setting. The purpose of the study was to investigate preservice teachers' experiences during a field-based mathematics methods course. The study examined the types of issues the preservice teachers found problematic, how those issues changed across and within settings, and how they resolved those issues. In the resolution of issues, the study examined the sources of authority to which the preservice teachers turned for information to aid in decision-making.

Background

The study was conducted during the fall quarter of the 1994-95 school year. The setting was a mathematics methods course for elementary education majors at the University of Georgia. The participants in the study were four female preservice elementary teachers, a classroom teacher, and a teacher educator (the author). During the 10 week term, the preservice teachers spent two hours two days a week in a local fourth-grade classroom observing the classroom teacher, interacting with children during group work, interviewing individual children, and teaching a small group of children.

The research questions that guided the study were:

1. What aspects of the teaching-learning environment of a fourth-grade classroom do preservice teachers find problematic?
2. How do preservice teachers try to make sense of what they find problematic?

Unlike prior studies that have found little change in preservice teachers' attitudes, beliefs, knowledge, or actions after a field experience, this study documented significant changes in the preservice teachers as a result of this field experience. One factor that may have contributed to these findings is the fact that the roles that the preservice teachers, the cooperating teacher, and I played in this study differed markedly from the traditional roles in field experiences.

The preservice teachers were not given specific responsibilities in the classroom until the last two weeks of the quarter. They were not asked to take attendance, take children to recess, assist individual children, or grade papers as is typically the case in more traditional field placements. This lack of demands on their time and effort left them free to focus exclusively on what was happening during the mathematics lesson. They also were not evaluated in any way during their time in the classroom, so there was no reason for them to be concerned about what my expectations or the classroom teacher's expectations were for their performance.

The classroom teacher assumed the role of a mentor and resource. She freely shared her ideas and materials with the preservice teachers and asked for nothing in return. The classroom teacher's ways of interacting with the preservice teachers closely paralleled her ways of interacting with her own students. In both cases, she assumed an investigative attitude and encouraged the students to generate their own solutions to problems.

As the teacher educator, I positioned myself as a learner in this experience. I told the preservice teachers, both in words and by my actions, that I was participating in this experience to learn about children and about preservice teachers. Although I was an experienced teacher, I did not pretend to have all of the answers to questions about teaching mathematics to children. I freely shared with them situations in both teacher education and teaching mathematics to children that I found problematic, and I revealed my thinking about these situations.

The classroom teacher assisted me in planning this study and was given the opportunity to provide input at every stage of the process. Her cooperation was vital to this study, and I wanted her to have the freedom to alter any aspect of it to meet her needs or the needs of her students. The fact that I did not simply send the preservice teachers to her classroom with instructions but rather came to her room along with them and was responsible for them during the class was unusual.

Methodology

Data were collected in the form of individual interviews, group discussions, individual journals, and video and audio tapes of the preservice teachers teaching. The data were analyzed from the perspective of the interpretive paradigm for teacher socialization (Zeichner & Gore, 1990). The interpretive approach involves an attempt to understand the nature of a social setting at the level of subjective experience. The purpose of the approach is to gain an understanding of the situation from the perspective of the participants and within their levels of consciousness and subjectivity.

The data were analyzed using the methods of grounded theory (Glaser & Strauss, 1967) and grounded interpretivism (Addison, 1989). The grounded theory method is a systematic inductive procedure for gathering and analyzing data for the purpose of generating a theory (Glaser & Strauss, 1967). Grounded theory and interpretive research methods are both constant comparative methods in which the researcher is constantly looking for and questioning "gaps, omissions, inconsistencies, misunderstandings, and not-yet understandings" (Addison, 1989, p. 41). Both methods also emphasize the importance of context and social structure in research settings, and in both methods data collection, coding, and analysis continue throughout the research process.

Findings

Preservice Teachers' Concerns. The specific issues that concerned the preservice teachers or captured their attention seemed to be somewhat cyclical in nature with the cycle repeating itself in each new setting (e.g., observing the teacher, interviewing children, teaching small groups). Their concerns seemed to closely parallel those identified by Fuller and Bown (1975) and by Schwab (1973). The first stage of concerns was identified by Fuller and Bown as classroom management and by Schwab as context. In general, the first issues that concerned the preservice teachers in this study were context-related, such as the children's behavior and engagement in the task at hand. They were concerned about both individual children and the group as a whole. These were their initial concerns when watching the classroom teacher and when teaching their own small groups.

The second stage identified by Fuller and Bown and by Schwab was concern with teaching. The issue that seemed to concern the preservice teachers was the teacher's role in the classroom and the execution of specific teaching actions. They were concerned about the types of questions to ask in order to elicit children's thinking and how to respond in a respectful manner to children's statements about their mathematical thinking.

Fuller and Bown and Schwab identified the third stage as concern for students. The issue that seemed to concern the preservice teachers was the children's learning. Their concerns manifested themselves in attempts to understand how or why a particular child or group of children arrived at a certain answer. Their concerns were also revealed in their attempts to describe children's thinking and to identify alternative meanings for the children's words. Additional evidence of their concern for children's learning comes from the remarks some of the preservice teachers made about the diversity of the students in the classroom. They were concerned about how to deal with diversity in children's levels of understanding without boring children who seem to understand or going too fast for children who do not seem to understand the content being taught.

The data suggest a contrast with the literature that postulates a stage-like array of concerns. The literature suggests that preservice teachers are initially very egocentric about their concerns about teaching (Fuller & Bown, 1975; Schwab, 1973). Only when they have resolved these concerns can they shift their consideration to the impact that their teaching is having on children. In this study, the preservice teachers did address egocentric concerns first, but they were able to address other concerns without having resolved their egocentric concerns. For example, as the preservice teachers struggled to make sense of the teacher's style and fit it into their belief system about mathematics and teaching, they noticed how much thinking and learning was going on in the classroom. Then they began to consider what this teaching style meant for a teacher. At that point, their concerns were primarily egocentric and centered on the fact this style of teaching necessarily leads to uncertainty for the teacher. They were especially concerned about the possibility that the children might introduce mathematics with which they were not familiar. They were also concerned that this type of teaching was harder on the teacher because it required the teacher to be "with it" all of the time. However, the preservice teachers were able to be objective enough to propose ways of dealing with the uncertainty and recognize the value for both the teacher and the students in this type of learning environment. It may have been the case that the preservice teachers were able to consider a range of egocentric and nonegocentric concerns about teaching because they were not responsible for teaching most of the time.

Sources of Authority. Belenky, Clinchy, Goldberger, and Tarule's (1986) descriptions of the ways in which women's voices can be expressed provide a way of thinking about the sources to which the preservice teachers turned for assistance in resolving educational dilemmas. In the beginning of this study, the preservice teachers tended to rely on the classroom teacher and me as the experts for answering their questions. They functioned primarily as received knowers, passively receiving whatever information the teacher and I gave them. The audiotapes of our earliest discussions after school are full of one sentence questions from the preservice teachers followed by five or ten minute soliloquies by the teacher and me. Rarely did the preservice teachers interrupt us to ask questions, and never did they question the validity of something we were saying. It was probably quite appropriate for the preservice teachers to seek information at that time as many of their questions pertained to procedural or factual aspects of the classroom or our opinions on some matter. There was no evidence that the preservice teachers were intimidated by us or that they did not feel comfortable talking to us. They were quite willing to recollect what they had seen in the classroom. They simply seemed genuinely interested in what the teacher, in particular, had to say.

As the quarter progressed, the preservice teachers became more assertive in positing their own ideas and opinions about things they had seen in the classroom or were thinking about on their own. They seemed to want to interject their opinions into the conversation. At first, they tended to treat everything that was said with equal validity and not comment on each other's ideas. At this point, they were functioning in ways similar to Belenky et al.'s subjective knowers. Their journal entries began to contain comments that began with phrases such as, "I think that . . ." followed by a statement of something that they believed to be true. Perhaps at this time, the preservice teachers were each more interested in their own ideas than in the ideas of others, as in the case of Belenky et al.'s subjective knowers, and were therefore content to leave other people's comments and even their own ideas unexplored.

By mid-quarter the preservice teachers were engaging in dialogues about each other's ideas and questions. At these times, the preservice teachers were functioning more like the procedural knowers described by Belenky et al. They often addressed questions to the teacher or me, but a peer would answer the question before we had a chance to speak. In several sections of the audiotapes of our later discussion sessions, the discussions are carried entirely by the preservice teachers. During these discussions, the preservice teachers sometimes openly considered the role that their prior knowledge was playing in

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their development of new knowledge about teaching mathematics. They reflected on their own learning experiences and the beliefs that resulted from that learning.

By the end of the study, the preservice teachers were beginning to take their own beliefs and knowledge, along with their experiences and the views of others, and to reconstruct their own ideas about teaching. Sometimes they shared these tentative ideas with the rest of the group, and at other times they just mentioned the ideas to me or wrote them in their journals.

Conclusions

Traditional field experiences are often organized so that the preservice teachers are busy at all times. They are often engaged in producing bulletin boards, copying worksheets, or grading papers. Such an organization leaves little time for reflective thought and careful consideration of classroom events. This study was intentionally designed so that the preservice teachers did not have any responsibilities in the classroom until the last two weeks of the study. At that time their responsibilities were strictly instructional. At no time were the preservice teachers asked to assume noninstructional responsibilities. During the first eight weeks of the study, they were expected to attend to and make note of any aspects of classroom life that they found interesting or problematic. They were then given opportunities to share these notes with their peers, the classroom teacher, and me. The organization of this experience and distribution of responsibility probably contributed to the preservice teachers' tendencies to attend to a wide range of classroom concerns and reflect on them.

Preservice teachers need opportunities to talk with classroom teachers, teacher educators, and their peers in order to gain additional perspectives on the problems of classroom practice. Traditional field experiences build in little time for communication with teachers, teacher educators, or peers. When opportunities for communication do arise, they are usually consumed by immediate concerns such as the schedule for the next day or a formal assessment of a lesson just taught. In this study, conferences with the preservice teachers, the classroom teacher, and me were scheduled as a significant component of the field experience. These conferences were planned in advance, and everyone was expected to attend. The preservice teachers were generally eager to attend these conferences, even though it meant commuting from the university to the elementary school each afternoon. They were eager to have an opportunity to ask the classroom teacher questions and to get feedback from her, their peers, and me about their evolving notions about teaching and learning. Because these conferences were not evaluative in nature, the preservice teachers were comfortable expressing their ideas and raising questions.

The sense of community that developed among the four preservice teachers, the classroom teacher, and me contributed to the positive nature of the experience. We were able to have meaningful discussions about particular events during the lesson because all six of us were in the classroom together. Although we did not all interpret what we saw in the same ways, we were able to spend our time concentrating on analyzing what we saw without having to describe the situation for someone who was not there. Although tradition precludes placing more than one preservice teacher in a classroom for field experience, the preservice teachers found it beneficial to have a community of their peers with whom they could discuss their experiences. Through their common experiences, they were able to develop a sense of community among themselves and began to see each other as resources.

Collaborative design of field experiences by cooperating teachers and teacher educators can have mutually beneficial results. Cooperating teachers and teacher educators sometimes view each other as adversaries and complain that they are being constrained by the other party. By working together to design this field experience, the classroom teacher and I were able to achieve common goals. The classroom teacher was not left with total responsibility for a preservice teacher for four weeks with unclear expectations from a university instructor, as so often happens in more traditional field settings.

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A METHODOLOGY FOR RESEARCH ON MATHEMATICS TEACHER DEVELOPMENT: THE TEACHER DEVELOPMENT EXPERIMENT[1]

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The teacher development experiment is an emerging methodology for studying the development of prospective and practicing teachers of mathematics. The method is based on the fundamental principle of the constructivist teaching experiment, that researchers who are engaged in promoting development are well-positioned to study that development. The methodology is grounded in an emergent perspective on learning and builds on the methodologies of the classroom teaching experiment and the case study. The teacher development experiment provides a dual perspective on teacher development by coordinating analyses of individual and group development in an attempt to understand the social, affective, and cognitive components of teachers' development.

The success of the current mathematics education reform effort hinges on the effectiveness of efforts to provide education and support for teachers to prepare them to carry out the reform in their classrooms. Towards this end, a knowledge base is needed to guide the creation of novel and effective teacher education programs. This knowledge base must include identification of key aspects of teacher knowledge and skills (the goals of teacher education), useful frameworks to describe how such knowledge and skills develop, and useful models of interventions that can promote such development.

In order to generate the requisite knowledge base, new research methodologies are needed. The need is based on the following dilemma. Descriptive research on teaching and teacher development generates accounts of what is in place currently. However, the current reform in mathematics education requires not only the reinvention of mathematics teaching, but also the reinvention of mathematics teacher education Therefore, research must contribute to understanding a process that is largely unrealized at this time. We seem to have a "catch twenty-two;" we are unable to foster adequate teacher development because we don't sufficiently understand the developmental processes, and we are unable to understand the developmental processes because we do not have situations to observe where teachers are developing this expertise.

A methodology is emerging which shows promise of meeting this need. I refer to this methodology, the subject of this paper, as a "teacher development experiment" (TDE). This methodology for studying teacher development builds on the basic premise of the constructivist teaching experiment, that a team of knowledgeable and skillful researchers can study development by fostering development as part of a continuous cycle of analysis and interaction, allowing researchers to work at the edge of their evolving knowledge. The emerging TDE methodology integrates an adaptation of the case study with its adaptation of the teaching experiment in order to collect and coordinate individual and group data of teacher[2] development.

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[1] A more complete discussion of the teacher development experiment methodology will appear in an upcoming volume produced by the National Science Foundation Divsion of Reasearch on Teaching and Learning, A. Kelly, & R. Lesh (eds.), *Innovative research designs in mathematics and science education.*

The Emergent Perspective

The TDE builds directly on the "emergent perspective" articulated by Cobb and his colleagues (Cobb & Bauersfeld, 1995; Cobb & Yackel, in press). From the emergent perspective, learning can be seen as both an individual constructive process (psychological) and a social process of the group.

> The basic relation posited between students' constructive activities and the classroom social processes in which they participate is one of reflexivity in which neither is given preeminence over the other. In this view, the students are considered to contribute to the evolving classroom mathematical practices as they reorganize their individual mathematical activities. Conversely, the ways in which they make these reorganizations are constrained by their participation in the evolving classroom practices. (Cobb, 1996, p.4)

Thus, the emergent perspective eschews debate over whether learning is primarily psychological or social. Rather it asserts the usefulness of coordinating the analyses that result from taking each perspective (psychological and social) as primary.

The Teacher Development Experiment Methodology

Guided by an emergent perspective, the TDE uses the developmental stance of the teaching experiment to study the different aspects of mathematics teacher development in the various sites where it occurs. This is pursued by coordinating whole class teaching experiments (in the context of teacher education) with adaptations of individual case studies. Researchers using the TDE methodology promote and study teacher development by using three key aspects of teacher development programs: courses in mathematics for teachers, courses in mathematics pedagogy, and classroom support/supervision. The collective development of participating teachers during coursework is studied through classroom teaching experiments (Cobb, 1996; McNeal, & Simon, 1994; Simon & Blume, 1994a; 1994b; 1996). The development of individual teachers is studied by adapting case study methodology to follow selected teachers through coursework and in their own classrooms. These different aspects of the TDE are described in the following sections.

TDE Teaching experiments. One major component of the TDE are whole class teaching experiments in mathematics courses for teachers and mathematics education classes. The researcher/teacher (teacher educator in this case) promotes development through the course activities. All class sessions are videotaped. The researcher/teacher meets with one or more of the other researchers on the team ("observers") between class sessions to analyze the previous session, generate and modify models of the teachers' conceptions and development ("ongoing analysis"), and plan the instructional activities for the next class session.

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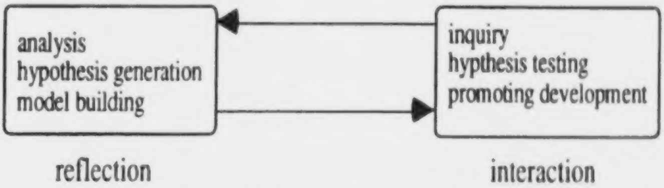
[2] In this article, "teacher" refers to both prospective and practicing teachers.

Whereas teaching experiments have generally been used to focus on mathematical development, the TDE classroom teaching experiments are used to generate models for the teachers' pedagogical development as well as mathematical development. This distinction is meant to communicate the breadth of the TDE focus, nonetheless recognizing that we can view mathematical and pedagogical development as interrelated. Teachers' pedagogical development (with respect to mathematics) may be influenced by their mathematical development, and conversely, their mathematical development may be enhanced as a result of pedagogical inquiry. Thus, the TDE teaching experiment makes use of the structure of its predecessors (Cobb & Steffe, 1983; Cobb, Yackel, & Wood, 1993) while broadening and changing the focus. Even in the mathematics courses for teachers, the focus is broader than in Cobb's (1996) classroom teaching experiments, because of the researchers' interest in how the teachers' concepts of mathematics teaching and learning evolve in the context of the mathematics courses.

The TDE retrospective analysis involves not only attempts to analyze data gathered over time, but also attempts to relate development from one course to the next, including mathematics courses and mathematics education courses. The models of teachers' development generated in the retrospective analysis have the quality of Cobb's (1996) paradigm cases, that is, they articulate ways of viewing teacher development that may be useful in considering other teacher development situations.

The advent of the constructivist teaching experiment (Steffe, 1983;1991) provided an alternative to existing research paradigms in which researchers are observers or quantifiers of either "natural" or "experimental" situations. The merged role of researcher and teacher provides an opportunity for the researcher to develop her knowledge through multiple iterations of a reflection-interaction cycle.



As part of the reflection-interaction cycle, the researcher/teacher brings knowledge shared by the research community and personal knowledge, including current hypotheses, to the interactions with the students. The researcher's interpretation of the interactions serves to support some aspects of her knowledge and to challenge other aspects, resulting in modification of that knowledge.

Case Studies of Individual Teachers

The TDE provides a dual perspective on teacher development by coordinating analyses of individual and group development. As discussed in the preceding paragraphs, the latter is accomplished through whole-class teaching experiments. The former is accomplished through an adaptation of the individual case study. However, both the teaching experiment component and the case study component involve coordination of social and psychological analyses. The whole-class teaching experiment involves looking at individuals' conceptions as well as the development of social practices. Likewise, the case study includes making sense of the social context within which individual development occurs, including courses for teachers and the classroom community of the mathematics class taught by the teacher.

The case studies of individual teachers that are included in the TDE make use of traditional case study methodology (Stake, 1978) but with a particular modification based on the TDE developmental stance (researchers promoting development). Extending the notion of fostering development as a context for studying development, TDE researchers take on the role of clinical supervisor in order to foster and study teachers' development in their own classrooms[3]. Simon & Schifter (1991) found that the role of "classroom consultant" (clinical supervision of teachers in their own classrooms) was a key component of fostering mathematics teacher development. The researcher uses the supervisor role to promote further development engaging in a cycle of reflection and interaction analogous to the cycle that characterizes the teaching experiment.

The researcher's role as clinical supervisor (based on Simon, 1989) involves regular observations of the teacher during the teacher's mathematics class and regular meetings with the teacher following those classes. Conversations may focus on the lesson; what came before or what will follow the lesson; the teacher's thinking prior to, during, and after the lesson (including the teacher's evaluation of the lesson); the mathematics involved, and the activity of individual students. The researcher also serves as a resource for the teacher by providing references for textual and instructional materials, ideas for lessons, and insights into aspects of the teaching-learning process. Further, the researcher serves as a support person for the teacher, a confidant for the teacher's sharing about emotional experiences that accompany engagement in radical professional change. Each of the aspects of the researcher's classroom supervision role contributes to the research team's ability to understand the social, affective and cognitive components of the teacher's development.

In contrast with the teaching experiment aspect of the TDE, the teacher sets the agenda for his interactions with the researcher in his own classroom, choosing how to make use of the researcher's support at any given time. Nonetheless, the researcher has her own evolving hypotheses as to what might contribute to the teacher's development. These hypotheses guide her responses to the teacher's agenda.

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[3] This aspect of the TDE can also be adapted to studying the development of prospective teachers by making use of their teaching during their practicum experience.

Data for the case studies include periodic videotaped observations of the teacher teaching mathematics. These observations serve as the focal point for interviews (audiotaped) with the teacher before and after each observation. The interviews explore the teacher's conceptions, motivations, and thinking with respect to the instructional decisions made prior to and during the lesson. Further understanding of the teacher's perspectives comes from inviting her interpretations of classroom interactions and individual students' behaviors. Data for the case studies of individual teachers are also gleaned from the TDE teaching experiments. Analyses of the individual teacher's participation in the teaching experiment classes are an essential part of understanding the teacher's development. (During class sessions, we organize the video taping of small groups to assure that the case study teachers appear regularly on camera.)

Conclusion

The TDE takes a stance in relation to mathematics teacher development that involves researchers in promoting teacher development in order to increase understanding of teacher development. It involves using the full extent of what is already understood about fostering teacher development as the starting point. Each subsequent intervention is guided additionally by what the researchers learned in the previous interactions with the teachers. Thus, the TDE is a tight spiral of hypothesis testing and hypothesis modification and generation.

The TDE allows researchers to generate increasingly powerful schemes for thinking about the development of teachers in the context of teacher education opportunities. The TDE takes as its object of study a teaching-learning complex that encompasses three levels of participants, the researchers, the teachers, and the teachers' students, and two levels of curriculum, the teacher education curriculum and the mathematics students' curriculum. By focusing on different aspects of this complex (indivisible) whole, one generates schemes about development (at three levels), schemes about teaching (at two levels), and schemes about curriculum (at two levels).

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WHAT ARE EFFECTIVE STRATEGIES FOR EVALUATION OF THE IMPACT OF LONG-TERM TEACHER ENHANCEMENT PROJECTS?

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It is increasingly apparent that if we want to impact change in how mathematics is taught at the elementary levels (K-6), we need to work with practicing teachers to update and improve their content and pedagogy knowledge and skills. This creates serious needs for increased capacity in mathematics at the school level. An emerging paradigm – the use of lead teachers or specialists – responds to an increased emphasis on local control and adaptation. Lead teachers are practicing classroom teachers who can assist other teachers in learning the necessary content and pedagogy in mathematics in order to improve the learning opportunities of students. Such teachers help spread ideas, facilitate communications, and plan and implement staff development.

As programs for lead teachers are developed and implemented, we need to document the impact of such professional development on the lead teachers themselves and, over time, on the *second tier* professional development and support provided by the lead teachers to their colleagues and peers. The central questions for this discussion group are:

1. What methods of evaluation and research (used as part of lead teacher projects) have been used successfully to evaluate changes in teachers' content and pedagogical knowledge and professional growth?
2. What are some of the thorny issues with respect to evaluation that remain problematic in addressing the needs to "measure" changes in teachers' content and pedagogical knowledge and professional growth?

Three large-scale teacher leader projects (Bright & Friel; Bush & Jones) focused at the K-8 levels provide the vehicles for considering a variety of evaluative and research-based methods for assessing impact and teacher change. Interventions were focused on teachers' content knowledge, pedagogical knowledge, and professional growth. Each of the intervention strands will be considered individually, interleaving evaluation and research methods conducted across the three projects. A matrix will be used to present summary information that itemizes methodology, and the anticipated and actual outcomes in terms of what was learned about impact of the project and teacher change. In addition, "gaps" or "holes" found in these projects in terms of methods needed to assess impact and teacher change will also be summarized.

USING CASE DISCUSSION TO EXPLORE THE WORK OF THE TEACHER EDUCATOR

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We propose a discussion session which will address the following questions: 1) In what ways can a case-based curriculum be used to explore mathematics and foster a spirit of inquiry into students' mathematical thinking? and 2) In what ways can teacher-educators use facilitator-support materials to develop a spirit of inquiry into their own practice?

Case discussions based on classroom narratives which present the mathematical thinking of students provide an effective medium for mathematics inservice work. Working from such cases, teachers explore mathematics for themselves, examine children's development of mathematical ideas, and learn about these through situated images of classrooms organized around student thinking. As part of our current professional-development project, teachers have had a regular assignment to write narratives of events from their own classrooms. As the body of narratives began to accumulate, we realized that they could be used as the basis of case discussions with other teachers. Now, furnished with over 400 cases, we have been selecting and organizing them to be used as the foundation of a coherent professional-development curriculum. This curriculum is currently being piloted at various sites throughout the country.

Along with the curriculum, we are also developing facilitator-support materials. One component of these materials is presented in the form of a journal written by a teacher educator, whom we call Maxine, reflecting on the events of each course session: how the activities went, the ideas and insights the teachers voiced, Maxine's concerns. Although Maxine is fictitious, "Maxine's journal" is based on actual experiences and events in an early pilot. In the same way that the cases present situated images of the mathematics classroom for teachers to work from, Maxine's journal offers situated images of an in-service course. It provides teacher educators an opportunity to reflect on the issues they face as they work to help teachers extend their own mathematical understandings and develop new notions of what it means to teach and to learn mathematics.

In the discussion group, participants will read and discuss the teacher-written case and then will turn to Maxine's journal to examine her thoughts and questions as she facilitated a group discussion on this same case. Maxine's reflections include her decisions as she worked to help her group of teachers shift their focus from their own teaching behaviors to their students' mathematical thinking. Finally we will reflect on the discussion itself: In what ways did the cases stimulate our own thinking? In what ways could such discussions support teacher-education/staff-development efforts? In what ways could narratives such as Maxine's journal help teacher educators conduct similar discussions?

STRUCTURAL SYMBOLIC INTERACTIONISM: CONCEPTUAL FRAMEWORK FOR UNDERSTANDING TEACHERS' INSTRUCTIONAL PRACTICES

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Structural Symbolic Interactionism [SSI] is a conceptual framework that can be used to integrate key concepts and data gathering procedures to investigate teachers' conceptions of their instructional practices and roles. There is no single universally accepted version of Symbolic Interactionism. However, most versions of Symbolic Interactionism are predicated on the following three premises. One, individuals act toward things on the basis of the meaning that the things have for them. Such things include everything an individual may note--physical objects, institutions, situations, and other people. Two, meanings arise out of social interaction. Three, meanings are modified during the social interactions of reflective individuals. (Blumer, 1969)

In SSI as in other versions of Symbolic Interactionism, individuals are thought of as applying names to situations, elements of situations, other people, and themselves in order to organize their behavior. SSI focuses on individuals' specialized conceptions of themselves that they construct during interactions in relatively stable social systems. These specialized conceptions are called roles. In relatively stable social systems, people can share common meanings of what is expected of individuals in recurring situations. Roles based on these expectations are important as they enter into an individual's process of interpretation. SSI highlights the reciprocity between individuals and society and that an individual's roles are negotiated rather than merely adopted. (Stryker, 1981)

In order to understand an individual's actions in a relatively stable social system, one needs to understand the individual's conceptions of his/her roles in the specific situation. An individual's role can be described in terms of: (a) the individual's goals regarding the function of the role, (b) his/her conception of the situation in which the role is performed, and (c) his/her conceptions of the expectations others identified as significant have for the individual (Stryker 1981).

The oral presentation of this paper will include an example of the use of SSI as a conceptual framework to understand differences in a teacher's instructional practice in a twelfth-grade Technical Mathematics course and in a twelfth-grade Precalculus course.

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CHANGING TEACHERS VIEWS OF MATHEMATICS EDUCATION RESEARCH: A CASE FOR COLLABORATION

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A collaborative research study between a secondary mathematics teacher implementing portfolios and university faculty resulted in the following: the teacher began to value and use mathematics education research, the university faculty gained insight into pragmatic issues in alternative assessment, the school district was assisted in planning performance assessment, and students reported benefits from compiling portfolios.

Teachers commonly report that traditional mathematics research does not inform their classroom instruction. They also comment that if research was in a *classroom ready* format, they would be more than willing to *try it*. However, since teachers are essentially responsible for classroom implementation and evaluation of goals described in the *Curriculum and Evaluation Standards for School Mathematics* (1989), it is critical that they come to value and use educational research to inform their classroom decisions.

In this qualitative research study, spanning two academic years, a secondary mathematics teacher implemented the use of portfolios. To allow her time necessary to make these changes and to observe her own classroom, university faculty taught classroom lessons, supplied her with current research articles and materials, interviewed her students, and served as a sounding board for her as she worked through dilemmas. The teacher, in turn, provided the researchers with access to the classroom, opportunities to obtain student artifacts for use in state-wide alternative assessment inservice. As a result, the university faculty were able to share these results with teachers across the state who were then able to see the possibilities for their own classrooms. The teacher also assumed the role of researcher as she reported her observations by writing and publishing an article about her experiences and by presenting a workshop at a regional mathematics teachers' conference. During this conference, the teacher talked about her experiences, distributed samples, and discussed research articles related to the project. This illustrated that research had become important for validating the time and effort necessary to change her classroom. She also assumed that other teachers would want research to support changing *their* classrooms.

As teachers become confident in their ability to analyze and critically implement research reports, they will not just blindly discard the *old* and implement the *new*, or just ignore the research, but will integrate what was good with what is better. This phase of reform comes from the classroom level. Collaborative research dispels the belief that research on effective teaching has very limited usefulness in the design and implementation of instruction and empowers all those who participate in these efforts.

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ELECTRONIC JOURNALS: A TOOL FOR ENHANCING STUDENT REFLECTION IN TEACHER PREPARATION PROGRAMS

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The national reform efforts in mathematics education are moving the mathematics education community to reforming undergraduate teacher education programs. These reform efforts include changing both content and pedagogy in the undergraduate elementary mathematics curriculum. Reflection is recognized as important to learning (Dewey, 1933; Han, 1995) and electronic journals provide a mechanism for enhancing reflection (Yan, Anderson, & Nelson, 1994).

This poster session addresses student reflections on mathematics content and methodology in a year-long mathematics content sequence of courses for prospective elementary school teachers through the use of daily electronic journals. The university is a participant in a National Science Foundation systemic teacher excellence preparation program, and these mathematics courses are two of many courses in the project. The researcher instructor studied one course each semester. The first course content included problem solving, numeration systems, and probability and statistics; the second course focused on geometry. Students established electronic mail (email) accounts at the start of each course and communicated with a small group of students and the instructor on a daily basis. The journal entries addressed content questions, beliefs and attitudes about learning and teaching, student concerns regarding the mathematics course, and reflections on what they were learning.

A pre- and post-survey recorded students' change over time concerning (a) mathematics as a way of knowing, (b) dislikes about mathematics, and (c) interests in mathematics. Data from interviews of three students conducted at the end of the academic year, daily journals between group members from the entire population (n=47) and the instructor, and the pre- and post-surveys from the entire population were coded and analyzed for keywords and signs of reflective activity.

The study showed that interactive email journals provided increased time for students to communicate mathematically, improved student confidence in mathematics, and afforded a new way of learning mathematics previously not possible.

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RESEARCH BASED PROFESSIONAL DEVELOPMENT MODEL: ACTION RESEARCH AND ALTERNATIVE ASSESSMENT

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Teacher education and professional development are seen as essential to educational reform efforts. The roles of teachers are recognized as pivotal to these reform efforts for it is teachers who change the curriculum, pedagogy, and assessment in the classroom (Dilworth & Imig, 1995).

The purpose of this paper is to describe a research based professional development model. Over the past several years, we have developed and revised the model in an interactive and constructive process of model building. The model with both cyclic and interactive processes, takes place among three basic components: (i) theoretical perspective; (ii) professional development experiences, and (iii) data gathering and analysis (Carter, Berenson, & Vidakovic, 1996). Beginning with the theoretical perspectives, we base our initial theoretical analysis on our understanding of learning theory (knowledge construction), teachers' content knowledge, teachers' pedagogical knowledge, and teachers' content pedagogy knowledge. This initial analysis informs the design of our professional development model experiences for teachers. The catalyst for change in our professional development model involves the teachers in using alternative assessments in investigations. Students' responses to alternative assessment items are collected and collaboratively analyzed and discussed in reflective interactions with other colleagues and teacher educators. Each professional development experience provides an opportunity for the researchers to gather data about the teachers' constructions. These analyses lead to the reformulation of either the theoretical perspective or the planning of the next professional development experience.

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TEACHER UNDERSTANDING OF STUDENT UNDERSTANDING

TEACHERS' ASSESSMENT OF STUDENTS' LEVELS OF THINKING AND MATHEMATICAL UNDERSTANDING

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One assumption underlying cognitively guided instruction (CGI) is that teachers will be able to learn how to assess the thinking of students and then use their knowledge of students' thinking to plan instruction. One part of assessment is deciding what evidence to attend to and what evidence to ignore in students' solutions to problems. This study was designed to determine what evidence 30, K-4 teachers found to be important in assessing the mathematical thinking of three first-grade students. The evidence teachers cited as important may be classified into seven categories: Use of manipulatives, understanding concepts and notation related to the problem, degree to which the problem solving process was completed, degree to which student needed teacher guidance, student's ability to describe a strategy, use of multiple solution strategies, and student self-concept.

Background

CGI has been repeatedly described and evaluated (Carpenter, Fennema, Peterson, Chiang, & Loef, 1989; Fennema, Carpenter, & Peterson, 1989; Fennema, Franke, Carpenter, & Carey, 1993; Peterson, Fennema, Carpenter, & Loef, 1989). Briefly, CGI is an approach to teaching mathematics in which children's knowledge is central to instructional decision-making. Teachers use research-based knowledge about individual students and then to adjust the level of content to match student performance.

The study was conducted at the beginning of a five-year teacher enhancement project (NSF Grant ESI-0940518) in which primary-grade teachers would be given opportunities to learn to use CGI as a basis of mathematics instruction. Teachers and mathematics educators from different regions in North Carolina formed five teams. Each team was composed of two teacher educators (i.e., team co-leaders) and six primary-grade teachers. The data reported here were baseline information against which comparisons could be made across the life of the project.

Method

Participants and Instrument

All project participants completed a transcript analysis instrument. The instrument contains three teacher-and-student dialogues (Mac, Tom, and Sue) that occurred while a group of 23 first-grade students worked individually on five written problems. The teacher interacted with Mac after he had completed the problem: *If frog's sandwiches cost 10 cents, and he had 15 sandwiches, how much did frog's sandwiches cost altogether?* As the teacher moved to Tom's desk, Tom was working on the same problem. The teacher's interaction with Sue as she was working on a different problem: *Frog had 15 sandwiches. If each sandwich cost five cents, how much do all the sandwiches cost altogether?* After reading the dialogues, participants were asked to state their conclusions about the three children's (a) level of thinking and (b) mathematical understanding. Participants were also asked to identify specific evidence from the transcript that was important to them in making those conclusions. No definition for the phrase "levels of thinking" was asked for or provided during the administration of this instrument.

Transcript analyses were completed during a morning session on the first day of the project's introductory workshop in May 1995. The instrument was administered in a whole group setting, but participants worked individually, without discussions. At this point in the project, all teacher participants (seven kindergarten teachers, eight first grade teachers, five second grade teachers, nine third grade teachers, and one fourth grade teacher) were classified as non-CGI teachers.

Analysis of Responses

Content analysis on verbatim written responses of the 30 K-4 teachers was completed manually. First, responses were dissected and the fragments grouped by content. Second, category labels were identified for either individual groups or clusters of related groups. The resulting seven categories summarized evidence that teachers used in determining students' levels of thinking and mathematical understanding. Each teacher's complete verbatim response was then re-analyzed within the framework of the seven categories. A grade-stratified grid was used to determine the percentage of teachers at each grade level citing each type of evidence.

Results

All teachers concluded that Mac and Tom exhibited higher levels of thinking and better mathematical understanding than Sue. No one assessed Tom at a higher level that Mac; about 25% assessed Mac and Tom at essentially the same level of thinking and understanding.

Most teachers defined "levels of thinking" either *relatively* as (a) higher level thinker versus lower level thinker (13 participants) or (b) advanced thinker versus less advanced thinker (six participants) or (c) *absolutely* in terms of Piaget's concrete to abstract levels (15 participants). Other terms included "good thinker", "independent thinker", and "cognition level", especially in the application versus knowledge level. The distinctions between higher level/lower level thinker and advanced thinker/less advanced thinker seemed to be a mix of (a) using higher order thinking skills (e.g., "It looks as though [Mac] relies on higher order thinking skills in order to solve problems.") and (b) being above versus on or below grade level (e.g., "[Mac's] level of thinking was at least third grade because he is already multiplying in his head mentally.") There was a consensus that use of concrete object necessarily indicated lower level thinking (17 participants) while use of mental math and visualization showed higher level thinking (nine participants).

Teachers were concerned about the degree to which students completed the problems. Specifically, teachers referred to students' ability to (a) understand the problem and develop a strategy for solving (22 participants), (b) complete one solution (five participants), or (c) identify additional solutions (16 participants). The amount of teacher guidance needed to complete a solution was cited as a major indicator of a student's level of thinking and mathematical understanding (14 participants), specifically focusing on students' explanations of solutions without intervention (12 participants). Teachers supported their ranking (Sue lower than Tom; Tom lower than Mac) by noting that Sue supported was unable to justify her solution, Tom provided a good explanation but had to be guided to the correct notation, and Mac provided good explanations of two solution strategies.

Teachers tended to focus on computational skills, citing use of multiplication as more advanced than counting or addition (12 participants). Mac's use of the shortcut, "writing a 0 after the 15" for the product of 15x10, was cited as a major indicator of higher order, advanced, or abstract thinking (14 participants). Money concepts, including money place value, meaning of units, and notation (i.e., 150¢ versus $1.50), were used to compare levels of thinking and mathematical understanding (23 participants). Some teachers cited Tom's expression, "one dollar and fifty cents," as indicating a lower level of understanding than Mac's answer of, "a dollar fifty." Teachers concluded that Mac understood money and the proper use of the dollar sign and placement of the decimal ([Mac] was able to see the 150 cents and 15 tens, 100+50, and $1.50."). Also, two teachers focused on student confidence and self-concept. One concluded that Mac had "math confidence" and Sue was insecure about her ability, while the other proposed that Sue wanted someone else to solve the problem because she lacked confidence in herself.

The evidence that teachers cited as important was classified as indicated in Table 1. Specifically, the concepts referred to in the second item are money concepts, and notation problems cited by teachers related to the use of dollar and cent signs.

Discussion

Teacher's attention to the use of manipulatives versus computational skills and mental math seems to be related to both a Piagetian concept of concrete to abstract levels of understanding and traditional curriculum design. Teacher responses indicate the use of manipulatives, computational procedure, and mental computation are recognized as "red-flag" clues of student understanding. These indicators appear to be used by teachers much in the same way some students use key words in problem solving. There appears to be no consideration that , in some cases, a use of a manipulatives might actually indicate a higher level of understanding than computation. Traditional curricula have emphasized symbols and procedures of symbol manipulation as indicators of mathematical understanding. Emphasis placed on students' proper use of dollar signs and decimal points in representing money is another indicator of teachers' concern with symbol sense as evidence of mathematical understanding.

Table 1

Percentage of Teachers, By Grade Level, Who Cited Each Evidence of Students' Levels of Thinking and Mathematical Understanding

Evidence	K (n=7)	1 (n=8)	2 (n=5)	3-4 (n=10)	All (n=30)
1. Use of manipulatives versus computational skill and mental math	71%	100%	80%	70%	80%
2. Understanding concepts and notation related to the problem	71%	100%	100%	80%	87%
3. Degree to which the problem solving process was completed	43%	63%	80%	40%	53%
4. Degree to which student needed teacher guidance	57%	63%	60%	40%	53%
5. Student's ability to describe a strategy and justify a solution	43%	13%	40%	40%	33%
6. Use of multiple solution strategies	43%	38%	80%	40%	47%
7. Student confidence and self-concept	29%	13%	0%	20%	17%

Teachers tended to focus on student' abilities to solve problems rather than on processes of problem solving. Specifically, teachers attended to the degree to which (a) problem solving is completed, (b) students need teacher guidance, (c) students can describe a strategy and justify a solution, and (d) students can use multiple solution strategies. It is interesting that no teacher attended to the difference between discourse with a student after the student has solved a problem as opposed to while a student is in the process of solving a problem. Mac had clearly finished solving the problem when the teacher began questioning him about his solution. The teacher appeared to fully understand and accept Mac's solution. Tom was still working on the problem when the teacher arrived at his desk; however, he was very close to a solution. Teacher discourse with Tom focused on helping Tom use correct money notation. Teacher discourse with Sue is substantially different from her discourse with the two boys. When the teacher began discourse with her, Sue appears to have been in a very early stage of problem solving. Without questioning, the teacher apparently assumes Sue is using cubes in a particular way. The teacher then enters a teacher-telling mode accepting responsibility for further defining what she thinks is Sue's solution strategy. She "drags" Sue to a solution., Sue seemed not to be following the teacher's strategy. The pattern of teacher discourse in these scenarios appears to be (a) listen to the student until a correct solution is described or until an error in strategy or notation occurs or until the wait time becomes long, (b) ask leading questions as long as student response is in line with teacher thinking, and (c) when the student fails to give the teacher-expected response, tell the student how to do the problem. It appears that this three-step pattern in teacher discourse with students might be so common as to not be viewed by teachers as either undesirable of remarkable. Although the teachers in this study did not seem to attend to important dynamics in the discourse scenarios, a few teachers did attend to student confidence as a problem solver and student self-concept. These teachers appeared to respond empathetically to the students, especially Sue, without attending to the teacher's role in affecting Sue-s self-concept.

The teachers in this study attended to evidence of student understanding that has traditionally been emphasized namely the correctness of answers given by the three students. When the specific teacher-expected answer was not given, the teacher switched the dialogue to a telling mode. This switch from an asking to telling mode appears to be so prevalent, yet, not one the 30 teachers who completed this instrument indicated recognition that it had occurred. It may be especially significant that the respondents did not suggest that the teacher in the dialogue may have failed to gain any insight into Sue's understanding of the problem. It may also be important to note that none of the respondents commented on the fact that Sue was working on a different, and perhaps more difficult problem

It is expected that teachers' assessment of student understanding must be accomplished within some framework of understanding about how students' think. In the absence of content-specific, research-based knowledge about children's thinking, teachers tend to rely upon some combination if general learning theories and current curriculum practices. The teachers in this study completed the instrument at the beginning of a five-year project in cognitively guided instruction,. As teachers progress through the project, they will study a variety of CGI identified problem-types and content-specific research results on student understanding. They will also develop skills in identifying and classifying solution strategies, and they will consider frameworks for planning instruction (e.g., sequencing problems) based on student understanding. As the project progresses, it will be of interest to determine if these teachers interpret dialogues in the transcript analysis instrument within new frameworks.

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FALSE IMPRESSIONS: LISTENING FOR UNDERSTANDING IN THE MATHEMATICS CLASSROOM

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Mathematics teaching and mathematics teachers have been the subject of intense scrutiny in recent years, both in North America and abroad. Many teachers who might previously have been described, especially by themselves, as favouring traditional teaching approaches, have begun to expect more from their students and from themselves, and have become more flexible in their teaching. Though it is widely considered that discussion work aids student understanding, this paper considers the notion that this open approach can deceive us into believing that articulate talk indicates real mathematical understanding.

The issue of communication in the classroom is at the heart of the current reform in mathematics education. In North America, the National Council of Teachers of Mathematics places a heavy emphasis on the importance of developing good communication skills (NCTM, 1989, 1991; Elliott, 1996). Encouraged by the wealth of literature propounding the advantages of problem solving activities and group work (NCTM, 1989, 1991; Slavin, 1983), many teachers have begun to adopt a more flexible approach to their teaching, and are now employing an increasing focus on discussion work in their classrooms.

Merely to put children into groups, however, and expect that they will learn mathematics is unrealistic. Richards (1991) cautions that "the mere saying of words does not make a conversation, nor does it make for communication" (p. 19), and as Pirie (1996) notes, communication needs a listener as well as a speaker. Davis (1994) suggests that in order to maximise the educational value of discussion in the classroom a change is needed in the way in which participants in the environment of the classroom listen to one another. In a notion also commented upon recently by Silver and Smith (1996), Davis notes that the ability to listen "depends upon a certain trust that the speaker has something to say" (1994, p. 280). Classroom observations suggest, however, that many teachers find it difficult to trust their students to be worthwhile speakers. Traditionally, it has been such teachers' belief that more mathematics will be learned if *they* remain the predominant speaker in the classroom; that learning must be initiated by them and mediated through them. Though they may purport to be sympathetic to the notion that they ought to be listening carefully to their students, evidence shows that such teachers frequently, in fact, listen to their students only to correct them (Davis, 1994).

While the current push to have students work in pairs or groups may be liberating, it can also have its pitfalls. Some students are articulate speakers and quick workers who dominate the discussions with other pupils and draw a disproportionate amount of the teacher's attention. Teachers often credit such students with superior understanding because they are more articulate. This conventional linking of understanding to articulation has prompted a mode of teaching (and attending) that privileges the articulate learner, however, this paper aims to demonstrate that mathematical understanding is not as tightly coupled to formulated speech as has been presumed.

The Study

The video episode described here is part of my ongoing research into the growth of students' mathematical understanding. The data were collected in a British high school at a time when I was a full-time teacher of mathematics, simultaneously engaged in a study of my own practice. Three pairs of Grade 7 and 8 students (corresponding to Grades 6 and 7 in North America) were video-taped during several weeks of mathematics lessons, and each student was then video-taped in a one-to-one interview with me at the end of the series of lessons. During the research project, which focused on students learning to formulate and manipulate algebraic expressions (Towers, 1994), the students were introduced to the notion of formulating algebraic expressions within the context of perimeter and area. The episode described here focuses on the attempts of two students, Kayleigh and Carrie (North American Grade 6), to find the perimeter of the shape shown in Figure 1. Working together, these students had earlier demonstrated their ability to cope easily with such a shape (again with missing values) when all the given information was numerical, not algebraic. When presented with this problem, Kayleigh immediately noticed that there was a missing value (the horizontal section) and articulated her problem as "We need to take c and b off twelve". Their conversation continued:

C: What happens if we go ten...
K: We don't know that length so we don't know what to do.
C: We do, 'cause that's twelve and then that's like what you said...oh...yeah, I see what you mean, see what you mean....oh, it's just...
K: Well, how are we supposed to know what c....what the value of c and b is?

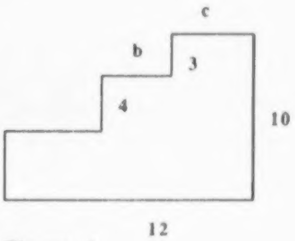

Figure 1

At this point both girls raised their hands to ask for help. Kayleigh articulated their problem to me as "We know we've got to take c and b from the value of twelve to get that [pointing to the horizontal line section with the missing value] but how do we write that 'cause we don't know the value of c and b to take away from twelve". Together we started to work on how that might be done, and Kayleigh seemed to realise that the three upper horizontal sections totalled twelve, but didn't use this information in the way I was expecting:

K: So if we make all that twelve there... [pointing to the three upper horizontal sections]
JT: It certainly is. It certainly is all twelve, isn't it? The c and the b....

I believed at this point that Kayleigh had 'seen' that she didn't need to use the b and the c at all, and that she simply needed to add two lots twelve, but Kayleigh interrupted:

K: Wait a minute, if we did twelve add ten add four add three is what, is....sixteen....nineteen....
K & C: twenty-nine.
K: So if we had twenty-nine add one…well, let's just say add twelve…no, that doesn't make sense does it? We need to make...add another twelve to...
JT: Twelve to where? Which bit's twelve?

Here, I had two reasons for asking this question. Firstly, I wanted to clarify for myself that Kayleigh really had realised that the three upper horizontal sections together were equal to twelve units, but also I hoped that by insisting that she consider this relationship again, Kayleigh might this time *hear* the significance of my continued interest in this feature of her explanation.

K: Along there [pointing to the three upper horizontal sections]
C: Along there 'cause if you'd pushed it all up....
K: And then we're going to take away c and b.

At this point I realised that Kayleigh knew that the upper three horizontal sections totalled twelve, but was only using this information as a means to work out the missing horizontal value, so I did not push the notion any further. After much further discussion, the girls, led by the more articulate Kayleigh, reached a final answer[1] of $41 - (1c + 1b) + 3 + 1b + 1c$.

During the one-to-one interviews, both girls were presented with the following two diagrams and asked to find the perimeter of each. In a manner consistent with the method she articulated in class, Kayleigh solved the first of these problems by finding the missing values on each of the sides and then totalling where appropriate to reach an answer of $14p + 20x$. This solution method was inadequate on the second of these problems, and Kayleigh initially could not solve it.

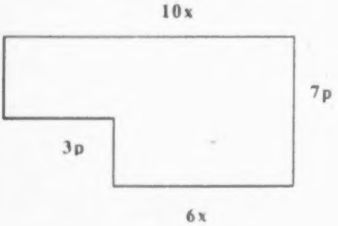

Figure 2

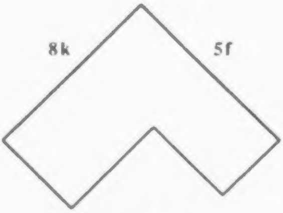

Figure 3

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[1] I discuss elsewhere issues surrounding the acceptance of this answer as a correct solution to the problem (Towers, forthcoming)

It should be noted that the question in Figure 3 was deliberately introduced in an attempt to discover whether any students were in fact using an alternative strategy, and to provide those who consistently worked out all the missing values with a reason to broaden their image. With help, Kayleigh saw that it was not necessary to know the values of the separate line sections and was able to produce an answer of $10f + 16k$. Carrie, on the other hand, wrote down a correct solution to the first problem ($20x + 14p$) in complete silence and so quickly that an observer is left with the distinct impression that she did not work out the missing values, but instead added two lots of $10x$ and two lots of $7p$. This impression is further reinforced by her rapid solution to the second problem ($16k + 10f$), again completed in silence, with no hint that she might have found this problem in any way a challenge.

Discussion

The issue for teaching is: Are there clues in the classroom interaction with these students that could, or should, have alerted the teacher to the possibility of one of the students (in this case the quieter, more reticent, Carrie) having an alternative strategy which had a broader applicability to other situations? The video, which is much more powerful in this instance than the transcript of Carrie's words, shows Carrie's initial excitement as she tries to articulate her ideas about what to do with the ten and the twelve, and her description of seeing the three horizontal sections as one line "if you'd pushed it all up", and then her boredom and detachment from the problem as Kayleigh and I worked together at length on finding a missing value (which Carrie may already have seen as superfluous).

My congratulatory tone when I acknowledged Kayleigh's 'seeing' that the three upper horizontal sections totalled twelve ("It certainly is all twelve") was missed by Kayleigh who was intent on getting across to me her intended method of solution. Kayleigh, if she was listening at all, appeared to take my words as affirmation of her method rather than as a clue to follow. This 'clue' may have been enough, however, for Carrie to be satisfied that her image for solving the problem, if, in fact, she did have a different image at that point, was viable. In fact, Carrie may have learned more by listening than did Kayleigh by insisting her ideas were heard[2].

Even in the short episode presented here it is possible to see how much the more articulate Kayleigh dominates the interaction when the teacher is present. Despite the fact that she consistently uses 'we' in explaining her approach, she interacts only with the teacher while the teacher is present, and it is noticeable that, from this point, Carrie takes very little part in reaching the solution. Whether, as the teacher, I contributed to this exclusion is debatable. It is natural for teachers, even when working with pairs or groups of students, to direct responses to questions and explanations to the student asking the

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[2] A more complete discussion of why Carrie may have chosen not to reveal her method is given elsewhere (Towers, forthcoming).

question or giving the explanation, as I did here. While I don't believe that I was guilty of listening 'only to correct', my listening, or rather not listening, to Carrie is a cause for concern.

Concluding Remarks

The example described here shows how difficult it is for teachers (even teachers who are aware enough about these issues to be taking the time to study their own practice) to really attend to listening to students in a busy classroom. My work suggests, however, that while it may be difficult it is an orientation which teachers must adopt if we are to create classroom cultures which are inclusive, and in which the voices are not competing, but conversing in fluid harmony (Davis, 1996). What is called for, then, is what Davis (1996) has called a movement beyond evaluative and interpretive modes of listening to a more participatory, hermeneutic listening.

In conclusion, and borrowing again from Davis (1996), the phrase *listening for understanding* in the title of this paper has two interpretations. Firstly, knowing what we are expecting to hear, we listen for, and credit students with, understanding that they may not have. Secondly, *listening for understanding* is also a reminder that listening is an aid to understanding, and that the adoption of a different approach to listening may promote greater understanding.

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DOCUMENTING OBSERVATIONS OF STUDENTS IN MATHEMATICS: A CASE STUDY

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The author collaborated with an eighth grade teacher in an investigation of the practicality of the "teacher s researcher" theory. The teacher experimented with ways to collect evidence of her students' mathematical thinking, and then make use of that information for different assessment purposes, such as instructional decision making, monitoring student progress, and grading. The core issue for the teacher became what to record, which led to the development to a master rubric and a recording strategy that made use of the rubric. The process of developing the rubric had a substantial effect on the teacher, enabling her to bring greater focus and coherence to her teaching.

In this paper I report on a year-long study of a mathematics teacher's attempts to document her observations of students as they worked in her class. The project was a collaborative effort to investigate whether or not it is feasible for a classroom teacher to keep a record of the kinds of assessment that are typically done informally in classrooms, namely, observations and interviews. Further, our goal was to determine how useful such information could be to the teacher for various assessment purposes, such as instructional decision making, monitoring student progress, and assigning grades.

Many recent assessment documents (e.g., NCTM, 1995; Stenmark, 1991) recommend that mathematics teachers make use of a wide variety of strategies for assessing students, including observations, interviews, portfolios, journals, and the like. Much of the evidence gathered from these sources is not easily quantifiable, suggesting a more interpretive approach. This view of assessment casts the teacher in a role not unlike that of an educational researcher, gathering data and making inferences based on the evidence at hand. There are varying interpretations of how this role of teacher as researcher might be played out in the classroom, from Mellin-Olsen's "practical hermeneutics" (1993) to Fruedenthal's naturalistic observations (van den Heuvel-Panhuizen, 1996). Regardless if the techniques advocated, teachers are being asked to make use of strategies for collecting data on student thinking, data that is usually qualitative and not quantitative in nature.

The act of gathering evidence and making inferences about student learning cannot be considered as separate from the other decisions and actions in which teachers are engaged. Assessment, rather than being viewed as isolated from the activities of teaching and learning, necessarily informs the links among teaching, learning, and the curriculum. The connections among these elements suggest that the use of alternative and more varied forms of evidence about students learning might have consequences for the teacher and the learner as well.

An eighth grade mathematics teacher (referred to in this paper as Ms. Vance) agreed to collaborate with me in an investigation of the feasibility of documenting the observations of students. Ms. Vance experimented with different strategies for recording what she observed in her two Algebra I classes. My role in this year-long study was to offer suggestions when solicited, and otherwise to systematically observe and analyze the experiment as it unfolded. The study was conducted with two classes of eighth grade Algebra I students in a public middle school in the Mid-Atlantic region of the U.S.. There were thirty students in one class and twenty-nine in the other.

The pedagogical approach in the Algebra I classes could be characterized by an emphasis on problem solving and on explaining solution strategies. Ms. Vance was quite clear that her primary instructional goal was problem solving, and this was apparent in her choice of curricular materials. She rarely used a textbook, bur designed the core of the curriculum around fifteen major tasks from such sources as New Standard and the PACKETS program from ETS. The focus in class was on finding ways to solve problem, often in collaboration with partners, and in sharing solution strategies. The latter was usually accomplished through whole class discussions.

During the first two months of school, Ms. Vance tried an assortment of methods for recording her observations of students. The first method was to simply write some notes to herself at the end of each class, This she quickly abandoned as unfeasible, when she found that three minutes' padding time between classes did not allow for such a high-concentration activity. She then tried to write notes and the end of the school day, but she found that most of the specific details of what she had observed were blurred at best or had disappeared entirely from memory by the end of the school day. It became apparent that she would need to find a method that could be used in class at the time actions were occurring.

When she found that writing took too much time and she tended to lose the slips of paper, she decided to try a tape recorder. Again she found that waiting until the end of class or the end of the day was not effective, so she began to bring it to class. During the first few days, while students were busy working in pairs, Ms. Vance would periodically pause to record what she was observing. Each time she turned on the tape recorder, the normal buzz of students would stop and every ear would strain to hear what she was saying. The students were obviously curious and perhaps a bit suspicious of what she might be saying about them. Ms. Vance explained what she was doing and let them listen to what she had said, which seemed to quell any anxieties. A few days later she decided to try letting the students have their turn with the tape recorder. As she circulated among the pairs, she would periodically ask them questions about their strategies or their conclusions and record these interview. As she moved from that group to another, she would add her own commentary of what she observed. This method proved to be popular with the students, who would often clamor to have their moment on tape. For awhile she did not abandon the written notes, but used a combination of notes and recordings, depending on her mood. Eventually she preferred the tape recorder exclusively, arguing that having it in hand was a good reminder to use it.

While the logistics of what method of documentation to use arose early, a more substantial issue became apparent when Ms. Vance reflected back in the early notes and recordings she had made. That was the issue of what to document. An analysis of her early records, both written and taped, show that they were largely unfocussed and not particularly useful for monitoring student progress. At the same time, it was clear that solving problems and explaining strategies were primary goals for these classes. This insight led Ms. Vance to develop a master rubric for the classes that encompassed components of "math power," focusing on problem solving, communication, and groups skills. To accompany the rubric she also developed a large poster-sized chart that listed the students along one axis and her master rubric across another axis. Whenever students were working on a extended task or project, Ms. Vance would circulate and record her observations and interviews. Later, with the chart in front of her, she listened to the tapes and recorded checks in boxes next to each students' name. The chart became a record of each students' progress in problem solving over the course of the year.

The primary use of information from the tape recordings and the chart was for monitoring student progress toward Ms. Vance's instructional goals. What Ms. Vance had developed was a means for gathering evidence about how well students were progressing towards her goals of problem solving, group skills, communication, and the more general "mathematical problem." By the end of the year she could retrace the progress they made from one task to another, or see general trends in their progress form one level to another. Because these same tasks formed the core of the students' portfolios, Ms. Vance decided that the information in the chart should be included in the portfolios, Together with the written work on each task, the portfolio became a record of each students' progress in problem solving during the year. Ms. Vance gave students an overall score on their portfolio, and this took the place of a final exam for Algebra I. In this was, the documented evidence also served the purpose of evaluating student achievement. Her great frustration was in having to quantify the portfolios into single letter grades at the end of the year.

In reflecting back on the school year, Ms. Vance felt strongly that the process of developing the rubric had been the most critical component in shaping her teaching. It gave her, she said, the opportunity to make clear (mostly to herself, and then to her students) what goals she had for her students. More that once she emphasized that, had she simply adopted a ready-made rubric from the start and used it to record observations without going through the process herself, the effect on her teaching would not have been as positive as it was. She said that articulating these goals in tun helped her internalize them, which brought greater focus to her teaching.

In some ways, this experiment was an investigation into the feasibility of putting theory into practice. The theory of instructional assessment, calling for teachers to engage in interpretive research, was put into the practical test of daily life in the classroom. What we discovered was a number of limitations in the extent to which teachers can play this role. First, the opportunities for documenting observations and interviews were limited to the type of instruction on a given day. It was not feasible for Ms. Vance to collect this kind of data except when students were at the initial stages of work on an extended task. Second, she had to choose a method that was practical and required little time to use (in her case, the tape recorder). Third, the critical factor in collecting data was knowing precisely what she was looking for. It was not until she developed her master rubric that the documentation process became coherent and useful to her.

This limited study gave only small indication of the possible positive consequences of a teacher's attempts to document student learning as she observed it in the classroom. More research is needed on the activities of teachers as researchers to determine the ultimate consequences for student learning.

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UNDERSTANDING STUDENTS' UNDERSTANDINGS THROUGH TECHNOLOGY-INTENSIVE TEACHER-CONDUCTED INTERVIEWS

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Research has suggested that mathematics instruction will improve if teachers are more sensitive to the nature of their students' mathematical thinking (Carpenter et al., 1989). Teachers may enhance their understanding of student understanding through in-depth content-based interviews of students. This study asks: What are the influences on and obstacles to teachers learning to use interviews to understand student understanding?

The targeted subjects for this report were three secondary mathematics teachers participating in CIME[1], focusing in part on understanding students' mathematical understandings in a technology-rich curriculum, *Computer-Intensive Algebra*[2] (CIA). During institutes in two consecutive summers, participants viewed videotaped student interviews, discussed techniques and purposes of interviews, designed and conducted student interviews, and analyzed interview data to assess student understanding. All subjects taught CIA during the intervening academic year. One subject also participated in an optional academic-year course that required her to interview her CIA students throughout the academic year. Primary data include transcribed tapes of their individual interviews with students and group discussions of these interviews.

Emerging influences and obstacles cluster around teachers' knowledge and conceptions in five different areas: learning/knowledge (e.g., nature of understanding, how it arises); teaching/schooling (e.g., belief about teaching); mathematics/curriculum (e.g., teachers' mathematical understandings; perceived importance of curriculum topics); students (e.g., perceptions of student traits and needs); and interviewing (e.g., purpose of interviewing, role of questioning). The extent to which a teacher reached the project's interviewing goals was influenced by the interactions among these five areas.

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[1] Computer-Intensive Mathematics Education was a multi-year teacher development and research project supported by the National Science Foundation under Grant No. TPE 91-55313. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation.

[2] Computer-Intensive Algebra (Fey et al., 1991) is currently being distributed as Concepts in Algebra: A Technological Approach (Fey et al., 1995).

TECHNOLOGY

A COMPUTER ALGEBRA SYSTEM (CAS) LEARNING STUDY: THE IMPACT OF ACCESS TO A CAS CALCULATOR ON THE NATURE OF MATHEMATICAL EXPLORATION BY PROSPECTIVE MATHEMATICS TEACHERS

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This study focused on the ways in which prospective mathematics teachers enrolled in a mathematics education course on using technology in the teaching of mathematics used a hand-held computer algebra system (TI-92). The study examined the nature of mathematical explorations conducted by these preservice secondary mathematics teachers as they used the TI-92.

The informed use of explorations in mathematics classrooms is integral to constructivist reforms in mathematics education. Individual and small group explorations enable students to learn mathematics in more personally and socially meaningful ways. In order to create appropriate explorations for students, teachers must understand what it means to explore mathematical ideas. Personal experience in learning mathematics through explorations can contribute to such understandings.

The term "mathematical exploration" has taken on a variety of meanings. For our purposes, the following characteristics were used to identify mathematical explorations:

1) The explorer generates and tests conjectures – the nature of this testing varies with the mathematical experience of the explorer;
2) Exploration is minimally directed – this is reflected in the nature of the tasks;
3) Exploration is learner initiated – there is a need to know;
4) Explorations have many possible directions and many possible outcomes;
5) Exploration involves developing mathematical ideas in that the goal of exploration is not necessarily known at the outset;
6) It is possible for exploration to yield an incorrect conclusion or no conclusion; and
7) Exploration is "doing mathematics."

Seminal work in the use of technology-based exploration in the learning of mathematics entailed observing students as they used dynamic geometry tools (Lampert, 1988; Yerushalmy & Chazan, 1990; Yerushalmy, Chazan, & Gordon, 1987, 1988). These studies have pointed out the impact of personal control of learning on the depth and nature of that learning. CAS calculators such as the Texas Instruments TI-92 now incorporate dynamic geometry tools and allow personal explorations that allow data importing from a computer-based laboratory (e.g., Texas Instruments CBL) or dynamic geometry tool into the calculator's numeric tools. In addition, the availability of symbolic manipulation capability enables students to conduct explorations (e.g., exploring a family of functions) in which they produce and manipulate symbolic, as well as graphic and numeric representations.

The study of explorations in mathematics is also informed by science education research pertaining to exploration. Roth (1995) states that "phenomenological representations are those that have been constructed directly from the experience" (p. 54) of the learner, and that this knowledge can be enhanced by proper use of technology. A number of efforts to facilitate explorations using technology are being developed (Pfister and Laws, 1995; Plano, et al., 1994; Tinker, 1994).

Our research has focused on ways in which prospective secondary mathematics teachers, equipped with these hand-held computer algebra systems, use such technology to explore mathematical ideas.[1] This study addressed the following research questions:

1) What is the nature of mathematical explorations conducted by preservice secondary mathematics teachers in the presence of a hand-held computer algebra system (TI-92), and in what ways do they use that technology in their explorations?
2) What are the barriers to and opportunities for symbolic reasoning that emerge when preservice secondary mathematics teachers use a hand-held computer algebra system (TI-92) for mathematical explorations?

Sample and Instructional Setting

Subjects for the study were eight of the 17 prospective secondary mathematics teachers enrolled in a mathematics education course during the spring semester of 1996 focusing on technology and the learning and teaching of mathematics (TLTM). All course participants had completed at least 18 mathematics credits at the calculus level and beyond. The eight students targeted for interviews and observation during class were selected to represent a range of mathematical abilities (mathematics GPA) and tendencies to explore (performance on a nonroutine mathematics task assigned the first class).

The TLTM course consisted of fifteen three-hour class sessions. A key component of the course centered on student explorations in which they used the TI-92 to learn mathematics and reflected on the nature of learning mathematics in technology-intensive environments. Mathematical topics were selected to reflect five mathematical themes suggested by Steen (1990): pattern, quantity, shape, chance and change. In particular, classes focused, in part, on mathematical topics related to one or more of the following themes: families of functions, mathematical modeling, symbolic reasoning, generating and testing data-based conjectures about geometric relationships, fractals, iteration, and/or conceptual underpinnings of calculus. In the course of their study of these mathematical topics students used, in addition to the TI-92 calculator loaned to them for their in-class and out-of-class use throughout the semester, spreadsheets, a computer algebra system (Child, 1993), and a dynamic geometry tool (Jackiw, 1991). During their explorations they created technology-generated graphs, used the CAS to perform symbolic manipulation, constructed spreadsheets, and conducted physical and dynamic geometry experiments (including those using a CBL) and fit curves to the data they collected.

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[1] This research was supported by an equipment loan from Texas Instruments and funding from the National Science Foundation (GER-9454048). Opinions, findings, conclusions and recommendations are those of

Pilot Study

In a semester-long pilot the course content was developed around the five themes, tested with a class of 13 students enrolled in the Fall 1995 TLTM course, and revised for use during the subsequent semester. Each interview task was piloted in one of two hour-long interviews with the 13 TLTM students.

The research team consisted of four graduate students, two mathematics education faculty members, and one mathematics faculty member. One of the mathematics education faculty members was the course instructor for the pilot study and the other was the course instructor for the study itself. These two faculty members, the most experienced in qualitative methods and observation techniques, provided for the others the necessary training in observation techniques, development of interview tasks and probes, and transcript production. Approximately four hours per week for 15 weeks were spent in these training sessions.

Data Collection

Major sources of data were classroom observations and individual task-based interviews. The written data for the study included verbatim transcripts of classroom observations and task-based interviews, as well as copies of three secondary data materials: student journals, examinations, and written assignments.

Classroom Observations of Explorations

Prior to each class the instructor identified portions of the class focused on mathematical explorations, and the four graduate student observers took field notes and audio-taped the target students as they worked alone or in groups on those explorations. The amount of class time devoted to explorations typically ranged from one-half hour to two hours. The observers' role was to be neutral and to interact with the subjects only to obtain clarification of actions or calculator key presses. A research team member also video-taped and audio-taped each class in its entirety.

From their field notes and audio-tapes the observers constructed transcripts within two days of their observations. The transcripts included target students' uses of technology and discussions between and among target students, their peers, and the instructor.

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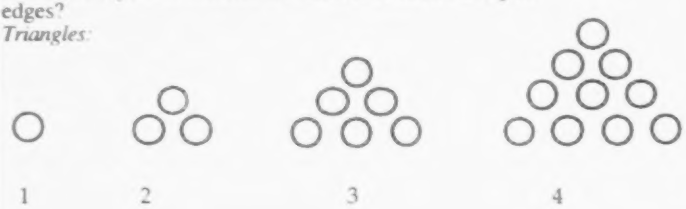
the authors and do not necessarily reflect views of the National Science Foundation or Texas Instruments.

Interviews

The researchers conducted one-hour interviews with the eight subjects during the second, ninth, thirteenth and fifteenth weeks of the course. Interviewers and interviewees were assigned on the basis of mutually available interview times.

The research team constructed task-based interview schedules using tasks that had been piloted the previous semester. The interview tasks were designed or chosen for their potential to engage students in technology-based mathematical explorations and were amenable to solution through use of the TI-92. Each of the first three interviews centered on two non-routine mathematical explorations. These included two "family of functions" tasks, two geometry tasks, and two mathematical modeling tasks. Figure 1 presents the six tasks used for the first three interviews. Tasks varied for interview 4 because they were tailored to subjects' responses in previous interviews. All four interviews were partially structured in that tasks were fixed and a variety of probes were available; however, during the interview probes were chosen to address subjects' responses and their explorations.

Each interview was conducted and audio-taped by one member of the project staff and video-taped by a second. The interviewee had access to paper, pen, ruler, graph paper, and a TI-92 and its manual. For ease of video-taping, the TI-92 that the interviewee used was connected to an overhead projection view screen so the image could be projected onto a screen behind the interviewee. The camera operator was thereby able to focus on the interviewee's written work (when the TI-92 was not being used) and on a sufficiently large image of the resultant TI-92 screens as keys were being pressed during TI-92 use. Project team members produced verbatim transcripts of the interviews annotated with calculator screen content and descriptions of other interviewee actions during the interview.

Interview	Task
1	The figure below illustrates a sequence of patterns of triangles of circles. The patterns are labeled 1 through 4 for each pattern. The triangles shown have circles in their interiors to fill them, when necessary. For example, figure 4 has one circle in its interior and nine circles along its edges. Will there ever be a triangle in the sequence for which the number of circles in its interior equals or exceeds the number of circles along its edges? *Triangles:* 
	Explore the family of functions $f(x) = \frac{A}{x} + Bx^2 + C$.

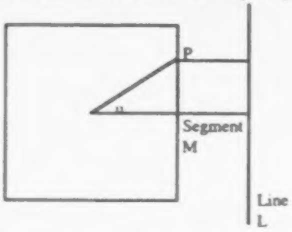
2	A square and a line L are sketched below. The line L is parallel to the side of the square. A segment, M, is constructed from the center of the square to L and perpendicular to L. Each ray drawn from the center of the square forms an angle, say u, with the ray M. Each such ray intersects the boundary of the square at a point, say P. A function f is defined which assigns to each angle u the shortest distance from the corresponding point P to the line L. Describe the behavior of the function f in as many ways as you can. 
	You can choose any of these tasks and do them in any order, but you must address all of the tasks using only the TI92. Tell the interviewer when you have read them all. Then talk about what you are thinking as you decide which one to do first using only the TI92. TASKS Draw a circle and measure or calculate the perimeters of various regular polygons inscribed in the circle. Describe your findings (verbally, symbolically, numerically or graphically or any combination). Find an equation for the value of the perimeter of an inscribed polygon as a function of its number of sides, using the circumference of the circle as a constant (say **c**). Find the polygon with the smallest number of sides whose perimeter is greater than 90% of the circumference of the circle. Which perimeter is a closer approximation to the circumference of a circle: the perimeter of an inscribed triangle or the perimeter of a circumscribed square?
3	Explore $f(x) = \sqrt{\frac{1}{ax^2+bx+c}}$ symbolically, numerically, and graphically.
	After being given a set of raw data representing instantaneous gasoline mileage for one car traveling at each of several different steady speeds over given stretches of road, interviewees were asked to investigate the relationships reflected in the data.

Figure 1. Descriptions of Tasks for Interviews 1, 2, and 3.

Data Analysis and Results

The project staff participated in an initial informal discussion about and analysis of each class immediately following its conclusion, and transcripts of small group sessions were constructed during the two days following the class session. These transcripts were discussed at the research team's weekly two to four hour data analysis sessions. A consistent focus of the discussions was an ongoing characterization of how the targeted students were using technology and symbolic reasoning in their mathematical explorations. Emerging results centered on the following: breadth of approaches that students with similar mathematics backgrounds take when conducting personal calculator-based mathematical explorations; relationships between familiarity with a CAS tool and the nature of student explorations; and the extent to which the tool influences the flexibility of the problem-solving approach. This study furthers our understanding of ways in which explorations are used in the learning of mathematics. As we observe students develop their abilities to use multi-faceted computing tools to explore mathematics, we will learn more about the impact of computing on mathematical thinking.

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DEVELOPMENT OF TURN AND TURN MEASUREMENT CONCEPTS IN A COMPUTER-BASED INSTRUCTIONAL UNIT

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This study investigated the development of turn and turn measurement concepts within a computer-based instructional unit. Turns were less salient for children than "forward" and "back" motions. Students evinced a progressive construction of imagery and concepts related to turns. They gained experience with physical rotations, especially rotations of their own bodies. In parallel, they gained limited knowledge of assigning numbers to certain turns, initially by establishing benchmarks. A synthesis of these two domains—turn-as-body-motion and turn-as-number—constituted a critical juncture in learning about turns for some students. Some common misconceptions, such as conceptualizing angle measure as a linear distance between two rays, were not in evidence. This supports the efficacy and usefulness of instructional activities such as those employed.

Numerous studies have addressed students' development of turn, angle, and turn measurement concepts, particularly in computer environments. These studies, however, have not focused on the *processes* of learning. We need to study the development of these concepts in the context of instructional units that reflect recommendations of recent reform documents. We are engaged in an NSF curriculum development project that emphasizes meaningful mathematical problems and depth rather than exposure (*Investigations in Number, Data, and Space*). One of the geometry units engages third-grade students in investigations of geometric paths. We investigated children's learning within this unit, emphasizing their developing ideas about turns and their measurement and the role of noncomputer and computer (our own version of Logo, *Geo-Logo*) activities.

The theories of the present study and the instructional unit are based on the notion that children's initial constructions of space emerge from action, rather than from passive "copying" of sensory data (Piaget & Inhelder, 1967). An implication is that noncomputer and Logo "turtle" activities designed to help children abstract the notion of path—a record or tracing of the movement of a point—provide a useful environment for developing their conceptualizations of simple two-dimensional geometric figures. Because turns (and angles) are critical to this view of figures, and because the intrinsic geometry of paths is closely related to real world experiences such as walking, path activities may be especially efficacious in developing students' conceptualizations of turn and turn measurement.

After working in Logo contexts designed to address ideas of angle and turn, children develop more mathematically correct, coherent, and abstract ideas about these concepts. This may be because they use a more personally meaningful turtle perspective scheme, instead of, for example, a protractor scheme (Clements & Battista, 1990; Kieran, 1986). Such benefits are educationally significant because students have considerable difficulty with angle, angle measure, and rotation concepts and because these concepts are central to the development of geometric knowledge (Clements & Battista, 1992).

Our goal in the present study was to investigate third grader's development of turn and turn measurement concepts within an instructional unit on geometric paths, including the role of noncomputer and computer activities in that development. Note that we use the concept turn as the amount of rotation along a path (a "turtle," or differential geometric perspective); we did not, in our research or in the curriculum unit, address traditional (interior) angle measurement or the relationship between turtle turns and such angle measurement. In analyzing the data, six themes emerged. Four themes were related to specific conceptual and procedural knowledge in the domain of turns: concept of turn; right and left directionality (clockwise and counter-clockwise); turn measure, with an emphasis on building benchmarks as units; and combining turns. Two were overarching themes: the role of the computer environment in students' development of turn concepts, and the dialectical relationship between two cognitive schemes, extrinsic perspective and intrinsic perspective, in students' development of knowledge of turns. Space constraints allow us to highlight only a few of these themes and one case study student.

Method

Procedure

We investigated third graders' mathematical thinking in the context of a unit of instruction, *Turtle Paths* (Clements et al., 1995), in two different situations. In the first situation, a pilot study, a graduate assistant taught the unit to four children. We conducted interpretive case studies of each child, with the goal of making sense of the curriculum activities as experienced by the individual student (Gravemeijer, 1995). Such interpretive case studies serve similar research purposes as teaching experiments (Steffe & Cobb, 1988), but are more naturalistic. They also better serve the needs of immediate curriculum development. In the second situation, a field test, two of the authors taught the unit to two classes. Data collection included pre and post interviews, pre and post paper-and-pencil tests, interpretive case studies of two students, and whole-class observation.

Participants

Participants for the pilot study were two girls, Anne and Barb, and two boys, Charles and David, from a rural town, all 9 years of age. Participants in the field test were students in two third-grade classes of inner-city schools, 85% of whom were African-American and most of the remainder Caucasian. As was typical for the school, 80% of the students qualified for Chapter 1 assistance in mathematics. The two case-study students for the field test were Luke and Monica, both 9 years of age*Curriculum*

In the *Turtle Paths* unit, students explore paths and the motions that create them. They walk, describe, discuss, and give commands to create paths. The main goals regarding turns are to have the students (a) build up images of turn as physical rotation, a change in heading or orientation; (b) distinguish between smaller and larger turns (gross comparison); (c) construct and iterate units of turn; (d) estimate turn measures using certain units as benchmarks; and (e) recognize that different physical rotations can yield the same geometric effect (e.g., rt 210 and lt 150).

Results and Discussion

Concept of Turn

Turns and turn commands were not as salient for the children as were forward or backward movements. They were constructed as conceptual objects and discriminated from other concepts only over significant periods of time. For example, in the first lesson, before the teacher introduced the turtle, she asked the difference between turning along a "U"- and a "V"-shaped path on which the children had walked. Luke talked with nearby classmates only about the space in the middle, with no mention of the turning actions. Turn commands may be less salient than movement commands because their representation often involves not a single graphic object, but rather a relationship between two graphic objects. Of course, it is true that the turtle performs a motion for each command, and in *Geo-Logo*, turns were slow enough to be seen. After the action had taken place, however, the records of turn actions (bends in paths) were not as salient to students as records of forward movement (line segments).

In sum, the concept of "turn" and "turn left" must be differentiated, with the latter including which way to turn. Yet another significant concept is that of turn measure, which we address in the following section.

Turn Measure

The limited salience of turns, combined with less familiarity with turn measurement, may account for children's slow and uneven development of concepts for turn measure as well as for turns. Students often needed a figural object (i.e., something physical or graphical to count) to produce a number for a turn measure.

For example, in attempting to draw an equilateral triangle with the turtle, Luke typed in fd 80 for his first side. For the first turn, he said, "lt f," but changed his mind and typed rt 60. When that didn't have the desired effect, he said that he should have tried lt 60, but changed his mind again and typed in rt 90. He was still dissatisfied with the effect and said, "No, you have to go over," as he turned his hand to point to the screen in the area that he wanted the turtle to go. He saw that the turtle needed to turn more if the path were to continue in the desired direction. He then synthesized his ideas about physical turns and turn measurement and changed the command to rt 120. Satisfied, he entered fd 80 for the next side and again faced a decision regarding the turn. He tried rt 90 and then also entered rt 80, most likely trying the side length in desperation. He entered fd 80 for the last side and became frustrated that his triangle was not correct. He sat back in his chair and stared at the screen looking dejected. He knew which command was wrong, but he didn't know how to fix it. He finally replaced the rt 80 with a rt 30, which, concatenated with the rt 90, constructed an equilateral triangle.

On a subsequent attempt to draw an equilateral triangle, Luke entered 90 for all fd, rt, and lt inputs. The sequence of commands concluded with "lt 90 lt 90." The researcher asked him to stop entering commands so she could copy the commands into her field notes. Luke sat back in his chair and stared at the screen. He noticed that the turtle was facing the opposite direction. He stated, "I ended up the other way, too. Isn't that 180?" He then observed that the last two commands did combine to make 180. This was a significant insight for Luke; without enacting body turns, Luke projected his personal experience into the turtle context and connected his body rotation and the turtle's rotation to the measure of 180°. Now Luke had a quantity to measure and a benchmark to use for that measurement; before that time, he did not possess a unit with which to compare turns.

Combining Turns

Meaningful combinations of turns would be a strong indication of the construction of turn measure. Luke did not combine turn commands as frequently as he combined length commands and gave more of an indication that what he was combining was not rotations per se, but numbers without quantitative grounding. In one instance, he examined the last three commands he had entered, and asked if he could change the two rt 90's to a rt 180. The researcher asked him if he can combine all three (rt 90 rt 90 lt 30) and he replied, "210". He tried it and noticed that the turtle was not at the same heading. "I'll have to leave it." The researcher prompted, "Stand up and try it." Luke did so, acting out each turn in the correct direction. He then said, "Right 150…you have to subtract the 30." This supports our contention that Luke often operated on numbers (only; i.e., his initial mental sum of 210) rather than on quantities and that he could operate on both simultaneously if asked to perform body turns, and thus focus on the physical rotation.

Role of the Computer

The feedback generated by the computer was instrumental in motivating children to reflect on their notions of turn measurement. For example, Luke thought that entering fd 120 would close his equilateral triangle (when rt 120 fd 100 was needed). The power of the medium was in offering feedback that was different in kind from the feedback offered by computer-assisted instruction. *Geo-Logo* provided nonevaluative feedback consisting of the enactment of his own ideas. Luke was quite surprised that his initial command did not work and was motivated to find out why.

Geo-Logo feedback consists largely of the graphic results of running Logo code precisely, without human interpretation. This was important to the pilot girls who, seeing the difference between their own interpretations of their Logo code for an equilateral triangle and the Logo turtle's implementation of that code, immediately examined and altered the code. Giving commands to a noninterpretive agent, with thorough specification and detail, has been identified as an important advantage of computer use in facilitating the learning of mathematics (Johnson-Gentile, Clements, & Battista, 1994).

The *Geo-Logo* environment allowed Luke and others to easily try different turns. Compared to regular Logo, the commands that are the same stay the same and do not have to be re-entered; more important, those that are changed are not executed as an additional command affecting the graphics; instead, the figure changes to reflect the change in the commands. This helped students encode contrasts between different turn commands.

Further, many students would use *Geo-Logo* to explore iterations of turns (`rt 90 rt 90`; or `rt 30 rt 30 rt 30`), and thus build a (superordinate) unit for measuring rotation. As stated previously, when combining inputs to turn commands, students were less confident about the results than when combining inputs to `fd` and `bk` commands (Clements, Battista, Sarama, Swaminathan, & McMillen, in press). Most students initially gave no indication that they were combining rotational units, per se (i.e., quantities). Rather they appeared to operate on numbers (sans quantitative grounding), use the result of that numerical operation as the input to a new turn command, and check the turtle's resultant final heading. The benefit of the computer is that they received this immediate feedback and thus began to give quantitative meaning to this combining operation.

Implications

Turns and turn measure were not ascendant in these third-grade students' thinking about geometric figures. However, they did make significant discoveries and gains in this domain. If turns were to be integrated throughout the K-6 mathematics curriculum, difficulties may diminish and benefits increase. Our results imply that static representations of turns, no matter how cleverly designed, may not only be inadequate for children's learning, but may delay their development of dynamic concepts of turn and turn measurement by limiting their construction of notions of physical motion and the integration of this notion with geometric figures and numerical measures. Instead, teachers need to emphasize physical motions (especially turning one's own body, especially for this age student), records of the transformation (e.g., by holding out one arm to maintain the initial heading, then moving the other arm through the turn), reflection on these motions (Hoyles & Sutherland, 1989), and connections between such physical activities and other dynamic activities such as those using computer and paper. Further, students should be encouraged to return to the physical motions as an aid during such computer and paper activities (despite their not uncommon resistance). Emphasis throughout these activities should be given to the establishment of perceptually salient units of turn and their relationships. Finally, in concert with other researchers (Hoyles & Sutherland, 1989), we have found that activities such as those featured in the curriculum, featuring well-defined goals (often with pictorial representations), are critical in developing concepts of turn and turn measurement.

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WHAT *ARE* THE DATA? AN EXPLORATION OF THE USE OF VIDEO-RECORDING AS A DATA GATHERING TOOL IN THE MATHEMATICS CLASSROOM

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Video recordings are being used more and more in educational research, but when do they actually form our data and when merely facilitate its collection? Research investigating the phenomenon of 'folding back' in the growth of mathematical understanding is used to illustrate the strengths and weaknesses of video recordings in classrooms, as a research tool.

There has been increased interest in ethnographic research within the mathematics education community over the past decade leading to a rich and varied collection of insights into aspects of the mathematics classroom culture and its psychological impact on teaching and learning. Much of this research is based on classroom observation and a growing voice is being given to teachers, bringing to our attention their own, unmediated, personal perspectives. An expanding proportion of this research involves video-cameras in classrooms. If, as researchers and readers of research, we wish to understand the work of others, to evaluate the psychological implications that are claimed, to explore the relevance to our own environment of research done in other contexts and cultures then it is vital that we know precisely on what data studies are based. Not only do we need to examine the tools with which the data were gathered, but also the nature of the data themselves. This is not as simplistic a statement as it may seem, particularly with reference to video-data.

In any research, with every question we ask we create a bias in the data, making, albeit unconsciously, a decision to leave other questions unasked. All research is to some degree subjective. We see what interests us; we look with a purpose. The field notes we take are already an interpretation of the phenomenon that we study. We rationalise as best we can the value of the data we gather and the worthlessness or irrelevance of that which we do not. Video-recording has been claimed as a way to capture everything that is taking place in the classroom, thus allowing us to postpone that moment of focusing, of decision taking. Yet who we are, where we place the cameras, even the type of microphone that we use governs which data we will gather and which we will lose. What video tapes can do is give us the facility through which to re-visit the aspect of the classroom that we have recorded, granting us greater leisure to reflect on classroom events and pursue the answers we seek. This paper places video-recording, as a means of data gathering, under scrutiny, acknowledging its strengths, while exposing its weaknesses and illuminating the need for honesty, both with ourselves and with others, as to the true nature of the data we are analysing.

I intend to illustrate some of the variety of occasions when video-recording in a classroom can offer evidence for valuable analysis, through the examination of my current research interest, namely the occurrence of the phenomenon of 'folding back', and in particular 'collecting'[1], in the growth of students' mathematical understanding. It is not necessary, for the purposes of this paper, for the reader to understand these concepts in detail[2]. It suffices to know that, within the Pirie-Kieren theory for the dynamical growth of understanding[3], the process of mentally (or indeed physically) returning to earlier mathematical understandings with the intention of illuminating some current outer level problem is known as 'folding back'. Within this phenomenon we have distinguished different mental activities. That which has been termed 'collecting', is a process of knowing that one knows what one needs to know, but of needing to review it, and possibly adapt it, in the light of the new situation. We distinguish it from instant recall and also from the need to substantially work on a concept before it is applicable to the problem in hand. (Pirie et al. 1996)

Since folding back can only be observed by close attention to the activities and talk of students as they work at a mathematical problem, the examination of classroom video recordings would seem to be the least intrusive, yet most inclusive, way of studying the phenomenon, particularly if it is coupled with a study of the students' written work. All choices as to methods of data gathering must be dependent on ones research questions, so what were some of mine? Let us consider just two of my questions. Firstly:

> "Do different types of teacher interventions favour or discourage folding back?"[4]

Before deciding on what I need as data, a decision must be taken as to whether to use pre-existing categories of teacher interventions or whether to create my own. If the former, I need to identify occurrences of folding back and classify them according to the type of intervention. The videos, here, are no more my data than the original classrooms were. Certainly they allow me time to make considered judgements on whether an episode is or is not folding back. I can reflect and revisit the recorded scenes at will, but the videos themselves are simply a vehicle, as is the classroom, for *gathering* the data with which I shall work. The *data* are the numbers within the categories, which I shall manipulate, analyse and interpret.

The question, however, appears misleadingly simplistic. In reality, I do not know what would be fruitful ways of categorising the interventions in this instance. Identifying episodes of folding back is, comparatively speaking, unproblematic, but what is it about the surrounding environment and context that might be influencing their occurrence? To answer this, it is the videos themselves that I need to study. My methodology here is such that I can examine any suitable existing video tapes and gather further specific data as and where I see the need. The method is based on the notion of grounded categories emerging from the constant comparison of episodes identified through theoretical sampling[5]. It is a method which allows me to look for anything in the intervention that might be an influence, give it a category label and confirm or discount its relevance through examining other episodes. I might note tone of voice, class grouping, gender of teacher and student etc. I let what I see on the video suggest ideas to me. Here it *is* the video recordings that are my data. It is the tapes themselves that form the basis of my interpretations. I could not do this analysis any other way. It is essential, not merely luxurious, to be able to re-examine the episodes, what went before and what came after. It is at this point, too, that I can exploit one of the real strengths of video as data. I can have others watch the episodes and suggest categories. The focus of the categorising remains the same, but I benefit from my data being looked at through different eyes and experiences. The categories, of course, will always remain subjective, in that there are no 'right' labels but the analysis becomes more powerful and more revealing as the perspectives of others are brought to bear on my task. "Rich descriptions, that explore the meaning structure beyond what is immediately experienced, gain a dimension of depth." (van Manen, 1988)

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[1] This label was originally coined by Lyndon Martin, a DPhil student of mine studying the phenomenon of folding back

[2] See Pirie & Kieren 1991

[3] See Pirie & Kieren 1994 for fuller description

It is important for me to state here that I work exclusively with the videos at this stage, never with transcripts. I feel that it is important that we be totally honest when stating the nature of our data. Working with the tapes as opposed to transcripts is neither intrinsically better or worse, but it *is* different. There is a great deal of information available surrounding working with text-based data; indeed there exist specialist computer packages which code, file, group and retrieve texts to the enhancement their analysis. Such analysis is based within a history of other qualitative research and can thereby claim a certain reliability and validity. It is not true to say that new insights cannot be had from repeated reading of texts, just as they can from re-viewing videos, but it would seem that the written word is less seductive, and once categorised and analysed, we rarely return to the text for a completely fresh reading. One does have the possibility of revisiting the tapes after working with transcripts, but one must be aware of the danger that only what is looked for may be seen and what has been concluded, confirmed. With video data the task is harder. We have to deal with a greater range of sensory perceptions, most relevant of which, to me, are how the words are said and what actions accompany them. Is "a closing of the eyelid .. a twitch, a wink, or a conspiratorial communication"? asks Goldman-Segall (1993)"And in any case, is it in any way relevant to my research issue?" I must also ask. It is herein that one of the

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[4] I am fully aware that there is a multitude of other considerations to be taken account of in dealing with this and all other research questions, but it must be remembered that my purpose *in this paper* is to focus solely on the role of video recordings *as data*.

weaknesses of working with such data lies. There are as yet no accepted ways of analysis through qualitative methods. We, as a research community, must build our own history, and for this very reason we must be as explicit and as public as possible about what it is that we do. For me, analysis involves a lot of 'sit, look, think, look again', frequently followed by persuading others to look with me because, quite apart from the insights they have to offer, I personally find that having to articulate and argue for my perceptions helps to crystallise them into either useful or inappropriate descriptions. A weakness of video data is that one can watch hours of video which turns out to have no relevance to the purpose in hand. In brief, transcriptions are frequently easier to work with, and in some cases more appropriate, but there will always be a loss of data and the researcher must *consciously* address the relevance of this loss. In addition, those to whom we would communicate should be informed of the true nature of the data on which the analysis was based, and if it *was* video data, on how that analysis was conducted. (Incidentally, had my question been concerned with particular types of *verbal instructions* I might possibly have considered working with transcriptions. The ease of analysis might have outweighed the loss of tone of voice.)

To return to my actual research question above: at this stage anything may be relevant, and it is for this reason, that I have chosen to video the classroom rather than use audio tapes, hence I stay with the video tapes as data. Once the categories have begun to stabilise through saturation (Glaser and Strauss, 1967) or have been discarded as irrelevant, I am in a position to work with these now defined categories and their relation to the phenomenon of folding back. I work with a new set of data: the categories and incidence of folding back.

The second question I wish to explore is:

> "How do students decide what mathematics to 'collect' for any given task, particularly when they are problem solving, where the clues may be few?"

If, as was the case in this research, a new avenue of interest opens out of existing explorations, video data comes into its own. It was the efforts of Martin[6], to examine in detail and flesh out the phenomenon of folding back, that gave rise to the label 'collecting'. We had not gathered data with this notion in mind but the second major strength of video data is the ability it affords to return to what has been recorded and view it again with some different purpose or focus. Of course, as we will see, all that one needs is not necessarily to be found. From the already identified episodes of folding back, I now need to extract those which fall within our description of 'collecting'. To do this I return to the video data and re-examined each episode to classify it more precisely. I pull out the examples of 'collecting' and start to examine what led up to them. Once again, I do not have a clear idea

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[5] This has its roots in Glaser and Strauss' approach (1967). See Pirie(forthcoming) for detailed description.

[6] Lyndon Martin. See footnote 1.

of what I am looking for. I record all and anything that seems to have bearing on the student's decisions. I classify these tentative category descriptors by student, by mathematical topic, by working groups, by interventions - from both teachers and peers - I involve others, ... till patterns start to suggest themselves. Now I have some pointers, now I can gather fresh, specific data to tease out details, to confirm or destroy conjectures. One thing stands out, all my information comes from tapes of students working in small groups. I have no video tapes of students working alone for the reason that up till now the use of language has been my prime interest, and students need to be in small groups for me to have access to this facet of their learning. When they talk together I can glean clues as to their thought paths but what is the effect of such grouping on the 'collecting' process? Do students work in the same ways when working alone? How can I know? Two different paths of procedure are needed. I can ask students to 'think-aloud'[7] as they work alone through problems and I can video tape them as they do this. I will then be able to watch what it is they are doing, writing[8], recording as they reveal their thoughts. This fresh data can then be analysed alongside previous data. On the other hand, to illuminate further the thinking processes of students working in groups an interview would seem appropriate. What I am trying to get at, however, is not what they think about the problem after they have finished working on it, but what their thoughts are *as they think their way through*. My best approach is through a technique called stimulated recall[9]. This involves the participants viewing with me the tape of their original group working and my stopping the tape at what I consider critical points and asking them to recall what they were thinking at that moment. My pilot study confirms the seemingly sparse literature, that much - but certainly not all - is to be gained this way. What is also clear is that I cannot go back to students appearing on tapes taken weeks or months ago and hope to get much useful data.

One particular pair of grade 12 students, Simon and Ann (Pirie et al, forthcoming), will always remain an enigma. Ann seems to have an amazing way of filing and cross referencing her previous mathematical understandings, but she relies totally on Simon to come up with the actual mathematical labels before she is able to recall the mathematics. For example, when she has told him that they learned what they need for this problem "on that day when Miss wrote the question out on the board wrongly and I said..." Simon immediately knows that she is referring to a time *eighteen months ago*! He responds with "that was probability" and Ann is instantly able to start to think about how to apply it now. How would she have worked alone? We will never know. Both students have now left school of course. Sadly video taping does *not* capture everything! What is needed here is a new round of data gathering which I shall structure by videoing groups of students problem solving and using stimulated recall to try to tease out their thinking processes while 'collecting'.

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[7] See longer paper for a review of the literature on think-aloud protocols

[8] See longer paper for tips on video recording and students' written work

[9] See longer paper for review of work using stimulated recall

Other issues related to video data and their collection abound. A fuller paper will include a review of the work of other mathematics education researchers working with such data, and provide a review of the literature on both stimulated recall and the analysis of think-aloud protocols. The notion of ethics and ownership of recordings will be discussed, as this is something we have to work out communally, particularly since one of the strengths mentioned above can be extended to the viewing of data gathered by others for our own, different purposes. Data sharing has to be addressed, as video playback in presentations and on the internet becomes acceptable and possible.

A paper on the use of video data is not complete without considering the practical and pragmatic decisions that have to be taken in addition to those more theoretically based, and discussed above. The longer paper closes with a consideration of this aspect of the research[10], with mention of the decisions, some technical, that must be taken on what and how to make video recordings, with practical hints on both creating the most flexible video data and tips for making analysis easier.

I leave untouched the issue of reporting the results of analysis of video data. Currently we are restricted to writing on the printed page, thereby, by definition, loosing vital evidence - else why would we have chosen to use video in the first place? At presentations we can - and I shall - illustrate our analysis with video clips, but until the day when PhD theses can include compilation video tapes, I and my students must rely heavily on your belief that my interpretation comes fairly from the data.

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[10] I wish to put on record my gratitude to graduate students of mine, Donald Cudmore, Lai Leng Soh, and Jo Towers for the help they have given me in compiling this section of the paper.

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MULTIDISCIPLINARY RESEARCH PERSPECTIVES ON AN IMPLEMENTATION OF A COMPUTER-BASED MATHEMATICS INNOVATION

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Researchers are challenged to respond to issues that arise as practitioners work in the transformative process, including the use of technology, ways in which teachers reorganize their instruction and curriculum using technology, and effects of these factors on learning. Here we summarize four interrelated studies, each striving to understand—from a different perspective—a year-long implementation of one technology-based mathematics innovation by all the fourth grade teachers in an elementary school.

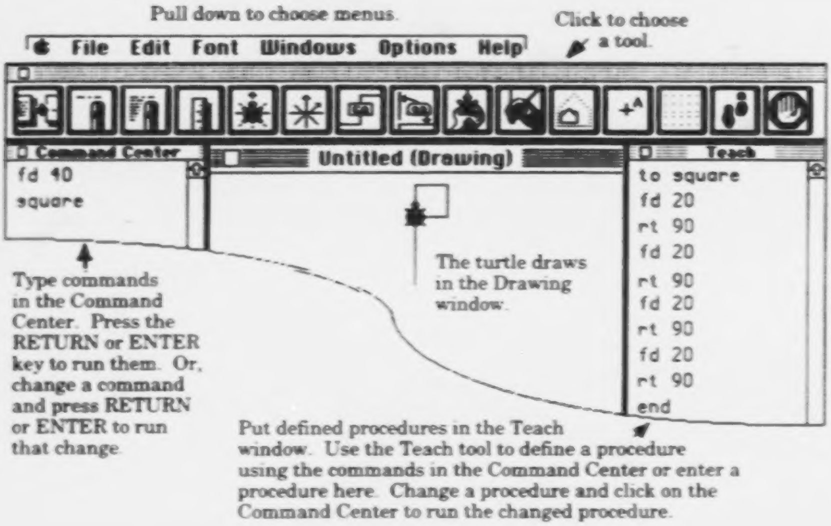
Given widespread attention to reform in mathematics education, NCTM's Research Advisory Committee has called for a transformative agenda for mathematics education, research that deals with what ought to be. Researchers are challenged to consider and respond to the problems and issues that arise as practitioners work in the transformative process, including the use of technology in classrooms, the ways in which teachers reorganize their instruction and their curriculum using technology, and the effects of these factors on children's learning. We conducted four interrelated studies, each striving to understand—from a different perspective—a year-long implementation of one technology-based mathematics innovation by the fourth grade teachers in an elementary school.

Design of *Turtle Math*

Our first project was a research and development effort involving Logo based software. One rationale for Logo programming is that students will learn mathematics by utilizing concepts that aid them in understanding and directing the Logo turtle's movements (based on constructivist and van Hielean theories, Clements & Battista, 1992). We reviewed research on this claim. This yielded mixed results; however, an analysis of these research findings provided significant guidance in the teaching and learning of geometry with Logo. On the basis, we designed *Turtle Math* (Clements & Meredith, 1994), a new version of Logo. We briefly describe the principles that guided the design of *Turtle Math* and some features of the software based on those principles (Clements & Sarama, 1995).

1 *Encourage construction of the abstract from the visual.*

Logo can help children construct mathematical strategies and conceptions out of their initial intuitions and visual approaches (Clements & Battista, 1992; Clements & Meredith, 1993). For example, there is a large literature on children's difficulty with turns and angles. Children can build more robust ideas of these concepts using Logo because they give turn commands and receive feedback. However, if children's Logo experience is not mediated, they can maintain misconceptions (Hoyles & Sutherland, 1989). *Turtle Math* provides several measurement tools; for example, an on-screen protractor, placed at the turtle's position and heading, measures turns. One arrowhead shows the turtle's heading. The other follows the cursor, which students move with the mouse. When they click this arrowhead "freezes" and a turn command is displayed. Rulers and other measurement tools are also available. Further, we should remember that even if students do not adopt our goals, and use visual and empirical strategies, the environment should continue to support their activity. The visually-oriented measurement tools allow students to approach task in a wider variety of ways. One of the main ways *Turtle Math* supports the growth of the abstract from the visual lies in its overall structure, described in the following section.



Pull down to choose menus.

Click to choose a tool.

```
fd 40
square
```

```
to square
fd 20
rt 90
fd 20
rt 90
fd 20
rt 90
fd 20
rt 90
end
```

Type commands in the Command Center. Press the RETURN or ENTER key to run them. Or, change a command and press RETURN or ENTER to run that change.

The turtle draws in the Drawing window.

Put defined procedures in the Teach window. Use the Teach tool to define a procedure using the commands in the Command Center or enter a procedure here. Change a procedure and click on the Command Center to run the changed procedure.

2. *Maintain close ties between representations*

The nature of programming creates the need to make relationships between symbols (code) and drawings explicit. But students may lose the psychological connection between the two when involved in long projects (Clements & Meredith, 1993; Clements & Sarama, 1995). In *Turtle Math*, students enter commands in "immediate mode" in a command window. Any change to commands in either location, once accepted, are reflected automatically in the drawing. The dynamic link between the commands in the command window and the geometry of the figure is critical. Also, the structure of the command window, long and narrow to the side of the graphics screen, instead of the traditional short but wide placement below this screen. This permits the immediate inspection of more commands, which facilitates connecting symbols and drawings, as well as pattern searching. Further, students can easily modify the code, encouraging experimentation and supporting later work with procedures.

3. *Facilitate examination and modification of code; encourage procedural thinking*

The environment should support easy creation, alteration, and use of procedures and highlight procedural-conceptual connections (Clements & Meredith, 1993; Clements & Sarama, 1995). The dynamic link also means that all commands in the command window represent a proleptic procedure. It can be defined as a procedure with a tool. A "Step" tool allows students to "walk through" commands to find errors or explore mathematical properties. Other palette tools allow easy editing and erasing.

4. Turtle Math *facilitates children's learning of mathematical ideas*

In several other ways, *Turtle Math* encourages students to build solid ideas about mathematics. For example, the Turn Rays option shows rays during turns. If you type rt 120, a ray is drawn to show the turtle's initial heading. Then as the turtle turns, another ray turns with it, showing the change in heading throughout the turn. A ray also marks every 30° of turn. *Turtle Math* also provides coordinate grids, scaling of distances (i.e., "fd 1" can mean forward 1 cm or forward 1 inch, allowing interesting use of fractions), geometric motions via menu, mouse control, and commands (including motions and scaling).

The role of teachers' beliefs

A separate team of researchers described how the teachers interacted with the innovation, with the goal of enhancing understanding of the dynamics between classrooms, schools, and innovations (Henry, 1995). This study examined teachers' beliefs about mathematics teaching, curriculum, and learning and the ways that these beliefs interacted with their implementation of a set of innovative curriculum materials, *Turtle Math*. Data were collected from a team of six fourth grade teachers at a suburban elementary school via audio-taped, semi-structured interviews, teacher journals, and field notes from classroom observations. The data were analyzed using a constructivist framework to determine how different teachers constructed implicit and explicit rationales and a process for implementing curricula in relation to his or her own personal beliefs and capabilities.

Teacher beliefs about the interaction between the innovation and their mathematics teaching were found to have profound effects on their implementation. One teacher, Carol, used *Turtle Math* extensively, due primarily to three factors, a) a pre-existing epistemology and pedagogy which were consistent with those underlying the innovation; b) a distinct need for the innovation due to her role as team leader and teacher of one of the advanced math classes; and c) a supportive research relationship. Another teacher, Beth, felt a need to use *Turtle Math* in her teaching but did not hold the antecedent beliefs or experience the support necessary to enable her to implement it fully. The other teachers in the team approached the use of *Turtle Math* as "one more thing" to do, and did not feel a need to pursue it, even after initial positive experiences for teachers and students.

Those involved in school reform may draw several implications from this research. Reformers should examine the correspondence between teacher expectations and the potentialities of a given innovation and be aware of the dynamics among teachers and research team members so that assignments are adjusted and support is allocated accordingly. School personnel must re-examine the relationship between curriculum, testing, ability grouping, and innovation, to motivate and empower teachers in their curricular decisions.

Role of the turtle metaphor in the development of mathematical ideas

A third team collected data from students and teachers assessing the degree to which the environment embodies the design principles in practice and the validity of the design principles; that is, whether the principles as implemented enhanced students' learning of mathematics in ways consistent with current mathematics reform recommendations. Results of both assessments were strongly positive with a few exceptions (Sarama, 1995).

Data was collected using both in-class observations and individual interactive interviews and were analyzed using qualitative methods. Results indicate that the *Turtle Math* environment had a positive effect on students' motivation to engage in most goal behaviors. Of those related to the six design principles, *Turtle Math* was neutral toward one, motivated three, and motivated and required two (see Table 1). Of the four main NCTM standards, *Turtle Math* motivated two (thought only partially for one) and motivated and required two. The motivation is intrinsic. That is, the software does not explicitly tell the students to engage in these behaviors, nor does it offer behavioral reinforcement. The students engage in these behaviors because doing so helps them meet the needs of the situation. Thus, evidence supports the general theme of a consistent philosophy in pedagogy and curriculum materials as well as the specific design principles and the realization of these principles in *Turtle Math*.

Genesis, variations and effects of cognitive dissonances in peer interactions

The fourth team examined student-student interactions from a socio-cognitive perspective (Swaminathan, 1995). The on-computer interactions of two pairs of grade 4 students from an suburban elementary classroom were video-taped during their math class for a year. Transcripts were coded according to the socio-cognitive and information-processing behaviors, and the SOLO taxonomy. Coded data were analyzed and interpreted using the analytic induction method to arrive at a descriptive and interpretive model of the processes and effects of cognitive dissonance.

The genesis of cognitive dissonances in peer interactions was studied from the perspective of the teacher, the curriculum and the children's interactional behaviors. The inquiries of the teacher instigated the children to explore extensions and to seek reasoning beyond the mere completion of the task. These explorations engendered cognitive dissonances within the children and established an ambiance of inquiry within the classroom. The innovative computer-based curriculum, with its structured activities and open-ended projects, prodded the children to think more intensely about their work and to comprehend it at more than just the visual level. The children's interactional styles and patterns of information processing determined the depth and cognitive value of their cognitive dissonances. Thus, the dissonances tended to vary according to the depth of reasoning and contextual processing evinced by the children. The effect of these dissonances on children's learning was evident in their heightened perception of certain mathematical concepts, enhanced reasoning abilities and improved use of strategies.

Final Words

Taken as a group, the studies generate several general conclusions and implications. The constructivist, van Hielean, and social-cognitive theories were supported. Further, curriculum innovation based on these theories and on empirical research can benefit students mathematical development. All but one of the six design principles were supported empirically. *Turtle Math* helped students integrate knowledge across topics and processes. It encouraged them to mathematize and explicate their actions and processes. Finally, it facilitated in one class, but not the other, engagement in mathematical projects. In sum, the students' mathematical activity was in line with the NCTM *Standards*.

Cognitive dissonance were numerous. They engendered the development of both of mathematics and problem-solving abilities.

Thus, the benefits seem clear. However, it was also palpable that implementing an innovation such as *Turtle Math* is demanding. One of the teachers, especially, however, benefitted from meeting that challenge. Carol became more interested in the specifics of their mathematical thinking and found new ways of assessing her students' mathematical understanding.

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Table 1: Motivation and facilitation for the goal behaviors related to each design principle of *Turtle Math*

	Motivate	Facilitate		
		mechanically	conceptually	application and extension beyond Turtle Math
Principle 1: abstract from visual	activities important for motivation	N/A	level of abstraction increased with experience	as abstraction increased, concepts were applied and extended
Principle 2: maintain close ties between representations	activities important for motivation	structure of command center dynamically links code to graphic	as link comes under conscious control (activities and teachers are catalysts)	scaffolded extension to off-computer situations
Principle 3: examination and modification of code	structure of command center and easy undoing encouraged modification	ease of undoing and structure of command center made this very easy mechanically	dynamic link supported students later de-bugging and editing	this was so common the students extended it to off-computer situations
Principle 4 procedural thinking	very little motivation from environments itself	Teach tool enables students to easily define procedures	teacher mediation is necessary, but accomplished with certain activities	not observed; dependent on teacher mediation and experience
Principle 5: freedom within constraints	open-endedness of software provided freedom; activities provided constraints	N/A	activities supported this principle. students became more creative with experience and knowledge	students extended activities and posed their own problem
Principle 6: consistent metaphor	wide range of activities and nature of software encourages metaphoric thinking.	N/A	turtle metaphor and different activities encouraged use of turtle metaphor for wide-range of topics	metaphor becomes strong enough to encourage use off-computer if situation evokes the turtle environment

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ROLE OF A COMPUTER MANIPULATIVE IN FOSTERING SPECIFIC PSYCHOLOGICAL/MATHEMATICAL PROCESSES

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We investigated the role of a computer manipulative in the development of children's geometric thinking. Kindergarten children (we focused on two case study children in a whole class setting) engaged in both on- and off-computer activities involving pattern blocks. The software, Shapes™, was designed to offer specific practical and pedagogical benefits as well as mathematical and psychological benefits. The children showed evidence of growth in their spatial thinking while working through the activities. Also, the data support the hypothesized benefits of the program. Children explicated their ideas about and actions involving motions only in the computer environment. Further, only on the computer did children differentiate between turns and flips.

We investigated children's geometric thinking as they were engaged in activities with pattern blocks (off-computer) and our *Shapes* software. The nature of the children's interactions was studied in an attempt to gain insight into the role on- and off-computer manipulatives can serve in the development of geometric ideas. Both the pedagogical/practical benefits and mathematical/psychological benefits were explored, as well as the childrens' conceptualization of angle, shape properties, and patterning.

The theoretical foundation for the study was our constructivist-based belief that both physical and computer manipulatives can aid mathematical activity, each in specific ways (Clements & McMillen, 1996). We believe that software design can and should have an explicit theoretical and empirical foundation, beyond its genesis in someone's intuitive grasp of children's learning, and it should interact with the ongoing development of theory and research—reaching toward the ideal of testing a psychological theory of children's mathematical development by testing the software that reflects the objects and processes of this theory (Clements, in preparation).

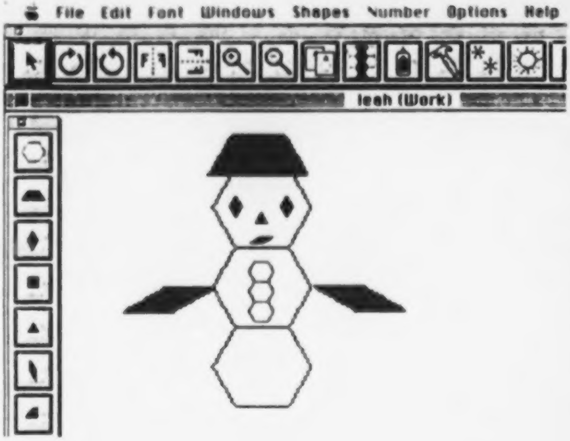
Method

Procedure

We investigated kindergartner's geometrical thinking using several activities developed with the *Shapes* software. Data collection included interpretive case studies of several students and whole-class observation.

What is Shapes*?—Description of the Software*

Shapes is a computer manipulative, a software version of pattern blocks, that extends what children can do with these shapes. Children create as many copies of each shape as they want and use computer tools to move, combine, and duplicate these shapes to make pictures and designs and to solve problems.



Hypothesized Benefits of *Shapes*

Shapes was designed on theoretical and research bases to provide children with specific benefits. These are described in the following sections.

Practical /Pedagogical Benefits

1. *Providing another medium, one that can store and retrieve configurations.* *Shapes* serves as another medium for building, especially one in which careful development can take place day after day (i.e., physical blocks have to be put away most of the time...on the computer, they can be saved and worked on again and again...and there's an infinite supply for all children).

2. *Providing a manageable, clean, flexible manipulative.* *Shapes* manipulatives are more manageable and "clean" than their physical counterparts. For example, they always snap into correct position even when filling an outline and—also unlike physical manipulatives—they stay where they're put. If children want them to stay where they're put no matter what, they can "freeze" them into position. The software manipulatives, then, often offer greater control and flexibility to children. They are just as meaningful as physical manipulatives and easier to use for learning [Clements, in press #1408].

3. *Providing an extensible manipulative.* Certain constructions are easier to make with the software than with physical manipulatives. For example, trying to build triangles from different classes. Is this possible, or must all triangles made be equilateral? If you can make triangles by partially occluding shapes with other shapes, many different types of triangles can be created. Making right angles by combining and occluding various shapes is a similar example.

4. *Recording and extending work.* The printouts make instant record-your-work, take-it-home, post-it paper copies. (Though we are also in favor of kids recording their work with templates and/or cut-outs, but this is time consuming and shouldn't need to be done all the time.)

5. *Building your own activities.* Teachers and children can create outline puzzles, "draw the other half" mirror challenges, and other similar activities easily and quickly.

Mathematical/Psychological Benefits

Perhaps the most powerful feature of the software is that the actions possible with the software embody the processes we want children to develop and internalize as mental processes.

1. *Bringing mathematical ideas and processes to conscious awareness.* The built-in turn and flip tools are a good way to bring geometric motions to an explicit level of awareness and explicate these motions.

2. *Changing the very nature of the manipulative. Shapes'* flexibility allows children to explore geometric figures in ways not available with physical shape sets. For example, children can change the size of the computer shapes, altering all shapes or only some.

3. *Allowing composition and decomposition processes. Shapes* encourages composition and decomposition of shapes. The glue tool allows children to glue shapes together to make composite units. The hammer tool allows the decomposition of those shapes. In addition, the hammer tool allows children to decompose one shape (e.g., a hexagon) into other shapes (e.g., two trapezoids), a process difficult to duplicate with physical manipulatives.

4. *Creating and operating on units of units.* In this way, *Shapes* allows the construction of units of units in children's tilings and linear patterns. A set of ungrouped objects, for example, can be turned together. However, in that case each shape turns separately. Only grouped shapes turn as a unit. Thus, the actions children perform on the computer are a reflection of the mental operations we wish to help children develop [Clements, in press #1408].

5. *Abstracting and extending patterns.* The pattern tool allows children to define a process for making tilings and other patterns and repeat that process. Children use the duplication tool to copy of shape or composite shape that they have constructed. They move this unit with the slide and turn tools. Then, they use pattern tool to repeat that duplication and transformation to create patterns in a more elegant, efficient way.

6. *Exploring symmetry dynamically.* The build-in mirrors allow children to explore symmetry dynamically. This is one time computer activity should actually precede activity with physical manipulatives. The *Shapes* mirrors are easier to use and understand than a physical mirror or Mira because the computer is doing the "reflecting." Further, the

software allows you to act on the shape and see the results unfolding *dynamically;* thus, symmetry can be seen as process rather than just static "reflection."

7. *Developing visualization and higher-level spatial representations.* All these tools and features help children develop visualization capabilities and construct higher-level spatial representations. Further the combination of these features are powerful mathematically: For example, children might build a shape that they think is symmetric by combining any number of other shapes and gluing them together. they can then duplicate this new shape, flip it and slide it over the original to check congruency. Again, these embody the mental processes that we want children to develop.

8. *Connecting space/geometry learning to number learning.* One of the most powerful benefits of the computer is to help children link their ideas and processes about number and arithmetic to their ideas about shape and space. Geometric models are used ubiquitously to teach about number. *Shapes* helps in that it dynamically links spatial and numerical representations in a wide variety of ways.

Findings

The data support the hypothesis that children would experience the benefits described previously. A discussion of each follows; space constraints allow us to provide only one example of each (a full qualitative report is in preparation).

Practical /Pedagogical Benefits

P1. Providing another medium, one that can store and retrieve configurations.

P2. Providing a manageable, clean, flexible manipulative.

When a group of children were working on a pattern with physical manipulatives, they wanted to move it slightly on the rug. Two girls (four hands) tried to keep the design together, but they were unsuccessful. Marisssa told Leah to fix the design. Leah tried, but in re-creating the design, she inserted two extra shapes and the pattern wasn't the same. The girls experienced considerable frustration at their inability to get their "old" design back. Had the children been able to save their design, or had they been able to move their design and keep the pieces together, their group project would have continued.

While working on the *Shapes* software, the children quickly learned to glue the shapes together and move them as a group when they needed more space to continue their designs. They also could leave them on the screen (or save the file) while they left to get a friend or teacher without fear of someone destroying their design.

P3. Providing an extensible manipulative

Matthew was taking his turn working with the physical manipulatives (off-computer) and was trying to fill in an outline of a man using all blue diamonds. At the end he was left with a space that only a green triangle would fit. He said, "If I was on computer I could make it all blue." Upon further questioning, it was revealed that the child knew that the two shapes were not the same, but that 2 green triangles is the same as one blue diamond, so half would go on the other shapes. The flatness of the screen allows for such "building up" and thus explorations of these types of relationships.

P4. Recording and extending work.

Carl, who always built extensive designs off computer, started a fairly complex pattern on computer. Soon several of his classmates were watching and giving him advice. They had to move on to another activity, but Carl's teacher saved the pattern for him to work on later. Carl, who was usually frustrated by having his designs destroyed was finally able to complete an extensive design. Marisssa and Leah's work, described previously, also provide support for this principle.

In sum, the hypothesized practical /pedagogical benefits were supported by the data, with one exception. No data was collected relevant to *P5*. Investigation of this hypothesized benefit will have to wait for further research (possibly with older students).

Mathematical/Psychological Benefits

M1. Bringing mathematical ideas and processes to conscious awareness.

When Mitchell worked off-computer, he quickly manipulated the pattern block pieces, resisting answering any questions as to his intent or his reasons. When he finally paused, a researcher asked him how he had mad a particular piece fit. He struggled with the answer and then finally said that he "turned it." When working on-computer he again seemed very sure of himself and quickly manipulated the shapes, avoiding answering the questions. However, he seemed more aware of his actions, in that when asked how many times he turned a particular piece, he said," Three," without hesitation.

M2. Changing the very nature of the manipulative.

The example of Matthew wanting to make an all blue man, recognizing that he could overlap the diamonds and be able to exactly cover a triangle space, also serves as to support this benefit as well.

M3. Allowing composition and decomposition processes.

Mitchell started making a hexagon out of triangles. After placing two, he counted with his finger on the screen around the center of the incomplete hexagon, imaging the other triangles. He announced that he will need four more. After placing the next one, he said, "Whoa! Now, three more!" Whereas off-computer, Mitchell had to check each placement with a physical hexagon, the intentional and deliberate actions on the computer lead him to form mental images (decomposing the hexagon imagistically) and predict each succeeding placement.

M4. Creating and operating on units of units.

M5. Abstracting and extending patterns.

When Monica finished making a unit for a pattern, she glued the pieces together and moved the group to the side. She then finished making a row of that unit and stated that now she had to "glue them." When asked how many she was going to have to glue, she pointed to each hexagon (a hexagon was the center piece of her unit) and counted to six. She built and operated on a unit of units as a unique entity. In many other cases, children glued to create units and referred to this as single entities (only in the computer environments).

M7. Developing visualization and higher-level spatial representations.

The example of Mitchell visualizing the triangles filling a hexagon applies here as well.

In sum, the hypothesized mathematical/psychological benefits were supported in those cases for which data were collected. The students in this study were not presented with the opportunity to engage in software activities relevant to benefits *M6* and *M8*. These children may benefit if provided with such activities, but these questions remain open for future research.

Final Words

The *Shapes* software was designed to offer specific practical/pedagogical and mathematical/psychological benefits. The data support the hypothesized benefits of the program. This also provides support for the underlying theory, in that testing the software provides one test of the hypothesized mental objects and processes in our psychological theory of children's mathematical development.

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CAN EDUCATIONAL COMPUTER GAMES HELP EDUCATORS LEARN ABOUT THE PSYCHOLOGY OF LEARNING MATHEMATICS IN CHILDREN?

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Many children don't like mathematics and find it boring. In this paper we present some of our observations and findings on children's psychology of learning mathematics in the context of computer-based mathematical game environments. Our findings point to some elements in educational computer games that satisfy children's learning needs and motivate them to learn mathematics. In this paper we list eight such elements that, we believe, play an important and synergistic role in children's learning. We feel that these games can help educators learn more about the needs of children when learning mathematics in general.

Introduction and background

Many students do not want to learn mathematics. They find it difficult to learn, irrelevant to their lives, and boring. In this paper we present some of our observations and findings about children's psychology of learning mathematics in the context of computer-based mathematical game (CBMG) environments. These observations and findings are drawn from an ongoing long-term research project aimed at investigating how to design game-based electronic learning environments for children (Sedighian & Klawe, 1996a, 1996b; Sedighian, 1996c). This research is part of E-GEMS (Electronic Games for Education in Math and Science), a collaborative university-industry-school project on educational computer games for children (Klawe, 1994). Our focus in this paper is on two closely inter-related issues regarding learning mathematics in the context of CBMG environments: 1) some of the needs of children when learning mathematics, and 2) elements that motivate children to learn the embedded mathematical content.

Motivation plays a central role in any learning activity (Dweck, 1986). Although there has been a great deal of research on motivation, there is "little scientific knowledge … about the factors that underlie motivation, enjoyment, and satisfaction" (Norman, 1993, p. 32). Malone (Malone, 1981) did a number of studies to investigate what makes non-educational computer games so captivating and fun. Since Malone's work, a number of researchers have suggested the use of electronic games to enhance children's motivation towards school subjects (Klawe, 1992; Norman, 1993).

"'Motivated'," Skemp (1986) states, "is a description we apply to behaviour which is directed towards satisfaction of some need" (p. 123). Some needs are innate while others are acquired. Whereas the need to eat is innate, the need to learn mathematics is acquired in order to satisfy other existing needs (ibid.). One of the most important aspects of motivating children to learn mathematics is to understand their needs. The need to know mathematics in order to create new technology or to understand other subjects is a remote one for children. This can be amended by placing children in situations in which learning mathematics becomes a tangible need. Playing games is a tangible need for children. When playing well-designed CBMGs (i.e., ones in which the mathematics is used as a continual and natural part of the game rather than as incidental diversions from the main activity), children gradually develop the need to learn the embedded mathematical content in order to satisfy their need to play the game.

Human beings use artifacts to extend their abilities to perform tasks. To assist children to learn mathematics, we can provide them with cognitive artifacts, i.e., artifacts that extend and improve their cognitive capabilities (Norman, 1993). Cognition, a synergistic partner of motivation, is another facet of learning. It should take place within an epistemologically meaningful context (Brown, et al., 1989). CBMGs can not only provide children with a context in which they find learning mathematics to be meaningful and useful, they can also provide researchers with a rich medium from which to gain new insights into the psychology of learning mathematics in children.

Research Method

Our research covers a period of over two years of close collaboration between several university researchers and almost 50 grade 6/7 students. Our research was conducted in different stages:

First, a number of prototype and commercial educational games were installed in the class. Kamran Sedighian visited the class weekly, from one to two hours, for over a year to observe the students playing the games, conduct informal class discussions, and audio-tape one-on-one interviews with the children. He also asked the students to keep journals of their computer activities.

Second, we designed Super Tangrams (ST) -- a very formal mathematical computer game aimed at helping grade six students learn two-dimensional transformation geometry. ST is considered by some mathematics educators as too challenging for this age range. The game involves the traditional tangram activity in which seven geometric shapes must be moved to fit together in an outline by using slides, turns and flips. It consists of a series of puzzles which progressively become more difficult. The design choices we made in developing ST were informed by the children's expectations of a fun and engaging learning environment.

Third, we installed ST in the class and visited the class weekly for almost six months to observe the students playing ST. In addition, we selected four students, considered by their teacher as weak achievers in mathematics, and videotaped several sessions of them playing the game with adult mediation. After approximately three months of playing ST on a weekly basis, we conducted and videotaped clinical interviews of 10 grade 6 students in order to qualitatively assess their understanding of the concepts of slide, turn and flip. Finally, a month after these interviews, all students had to write a 'surprise' test on concepts in transformation geometry.

Observations and Findings

The findings that we present below evolved in the context of observations and reflections we made throughout the two-year process described above. We strongly feel that the following observations and findings should be viewed collectively as a set of synergistic factors that affect children's learning of mathematics and not considered in isolation:

1) *Meaningful learning:* Computer games are an integral part of children's popular culture (Provenzo, 1991). We have found that situating mathematics learning in a computer game environment brings greater relevance to the subject for children. In our interviews with children many of them made comments such as "if you're doing it [mathematics] out of a book it's really boring, and you don't want to do it," whereas, "if you're doing it out of a game or something then you're wanting to do it and you're having fun with it so you can concentrate on what you're doing instead of just getting it over with and then forgetting about it 5 minutes later." We have found that CBMGs provide environments in which children find learning mathematics to be meaningful and useful. For instance, after playing ST, one student commented: "someone finally found a use for geometry."

2) *Goal:* Oftentimes CBMGs provide children with a goal or a set of goals to achieve. For instance, in ST the ultimate goal is to finish the game or, as children put it, to "beat the game." In addition, throughout the game there are also intermediate goals such as fitting a puzzle piece into place, finishing a puzzle, advancing to a higher level in the game, and increasing one's score. We have noticed that such goals create a sense of mission in children. Often the students begged us to allow them to stay in class during recess periods so that they could finish a puzzle. Many children expressed that they enjoyed CBMGs because there is a "goal that you've got to accomplish something and then you get excited after you've accomplished it" and that with a goal "you have something to look forward to." In contrast, children expressed a sense of frustration with school mathematics because it often involves completing a set of worksheet-type problems that do not provide them with meaningful goals -- "when you do it on paper all it is is solving a problem."

3) *Success:* Accomplishing the goals of CBMGs can provide children with a sense of success. For example, many students playing ST keenly kept track of their score and advancement in the game as measures of their success. A particularly interesting instance was when some slow-paced students became very excited upon reaching a score of 1000. Their excitement was not because their score was necessarily high compared to other students or that they had gained a sense of the scoring scheme of the game, but simply because 1000 was a large number to them and therefore signalized success.

In our research we have found that children want and need to be successful in the social environment of their class. They need to succeed in mathematics to want to learn more mathematics. Therefore, we should place them in environments of learning mathematics that provide this sense of success. An important factor in feeling successful is how children perceive their mistakes. We found that since children could recover from their mistakes in the game without forfeiting much they would not feel threatened by making mistakes. Rather, mistakes became stepping stones for later success in the game allowing children to progress at their own pace towards the ultimate goal.

4) *Challenge:* As children played CBMGs we often heard them say: "I like the challenge." To almost all of them being challenged in a game meant that they would not be bored. In contrast, they frequently referred to school mathematics as boring. We found that the degree of challenge children asked for corresponded to their individual abilities; they needed to face a challenge, but not one that was beyond what they could handle. They particularly liked games that would become progressively more challenging. They often became bored with game activities which were repetitive. As they mastered a certain level of difficulty, many children would want to immediately move on to a new challenge. Children need to be constantly challenged and seem to thrive on it. They enjoy learning mathematics in the context of a fun challenge.

5) *Cognitive artifact:* In our observations of children playing ST, we have identified two factors which make ST a cognitive artifact for them: 1) interactivity and 2) communication. Many students commented that they enjoyed the fact that they could interact with the mathematics in ST. For instance, one of them wrote in his journal that ST "is a good way to learn motion geometry instead of reading it from a text book". His reasons were, "One, you cannot move things in a text book. Two, a text book does not talk to you." The students were assisted in their understanding of transformation geometry by directly manipulating the embedded mathematical representations in the game and immediately seeing the consequences of their actions. This interactive learning process helps children develop a sense of the mathematics they are learning.

ST also provided students with a concrete, external reference point by which they could communicate their thoughts. We found that whereas students were often shy to discuss transformation geometry concepts away from the game, while playing the game those very same students were motivated to freely talk about transformation geometry and to ask questions. Moreover, in the videotaped interviews of 10 sixth graders, they all expressed their understanding of transformation geometry in terms of the game. For example, a student who was describing her understanding of rotation drew the exact representation of rotation given in ST and explained it in the following manner: "It has an arc; I don't know what you call it; and it has, I don't remember what the points are called, and then there is an arrow and there's a red dot and a green dot." Often students lack the proper vocabulary to discuss mathematical concepts, and, as a result, are shy or afraid to say the "wrong thing" in front of an adult or a teacher. Children need cognitive artifacts such as CBMGs to motivate and allow them to express their thoughts about mathematics, even if this expression is initially game-bound.

6) *Association through pleasure:* Children need to associate mathematics with some pleasant memory. This association assists the concepts to remain with the children. One of the students articulated this point when he made the following comment on how he remembers the mathematical concepts in one of the games:

> "You think about the sound and then you remember the game and you remember what you were doing in the game and it can help you. Or if you're doing a math problem and you don't know how to do it with like a math book because it's boring, but then you played it in a game and you remember it from say the color or sound just comes up in your mind and you can go 'Yeah, that's how you do it!'"

7) *Attraction:* Several students who were weak achievers in mathematics said that they were "obsessed" with ST and, if allowed, "would play it all day." A few said that they had even dreamt about it. Other students, who rarely showed initiative in learning mathematics, took ST puzzles home and figured them out over a weekend. We have found that CBMGs can create environments in which children get excited about the embedded mathematics and, therefore, are willing to be immersed in it and spend time learning it. We feel that in order to stimulate children to intensely think about mathematics, they need to be put in learning environments which attract them to mathematics and allow them to experience the joy of learning it.

8) *Sensory stimuli:* At the initial stage of our research several of the computer games which we installed for the children had minimal sensory stimuli. Many children were not particularly approving of these games because they had no fancy graphics, their images were in black and white, their animations were very simple, their sound effects were primitive, and they had no background music. We have found that for children such sensory stimuli add to the fun of playing the game and make the learning of mathematics more enjoyable and memorable -- as one of the students put it, "they add flavor to mathematics."

Conclusion

The main focus of our research is investigating the design factors of motivating electronic learning environments for children, particularly CBMGs. In the course of this research we have learned much about what motivates children, what sustains their interest, and how children's attitude towards mathematics can be improved. Although our findings and observations have been made in the context of CBMGs, we feel that they can shed some light on the psychology of children's learning of mathematics in general. These findings are by no means complete. Our hope is that they will encourage other researchers not only to do further studies on the learning of children in the context of CBMGs but also to investigate how to create other learning environments in which the above elements are cohesively and dynamically integrated.

Acknowledgments

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STRATEGIES CONSTRUCTED AND OBSTACLES ENCOUNTERED BY STUDENTS USING TI-92S IN FIRST-YEAR ALGEBRA

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First-year algebra students used a TI-92 symbolic calculator in a taught curriculum centered on function and mathematical modeling. Individual interviews with targeted students and classroom observations focused on students' strategies for using the tool to answer real-world questions about situations modeled by functions of one variable. Although they were taught particular calculator-based strategies, students were frequently unsuccessful in solving questions that the teacher and curriculum developers considered simple. The important part of the students' work is not their lack of success but rather the strategies they constructed and the obstacles they encountered in trying to use appropriate approaches.

First-year algebra students used a hand-held computer algebra system (Texas Instruments TI-92 symbolic calculator [TI-92]). This study focuses on how they used the tool to do mathematics and what obstacles kept students from successfully answering tasks within the Computer-Intensive Algebra curriculum (Fey et al., 1995).

Framework and Background

Numerous studies indicated that the use of four-function, fraction, scientific and graphics calculators in elementary and middle school need not obliterate acquisition of skills (Hembree & Dessart, 1992, 1986). Dunham and Dick (1994) summarized various ways in which graphics calculators can contribute to older students' understandings of function. Heid (1988), Palmiter (1991) and others collectively determined that computer algebra systems (CAS),use in calculus can facilitate understanding of function. Heid and Zbiek (1993) document similar results from the use of a CAS in a functions-oriented first-year high school algebra course.

The potential of graphics calculators and CAS may lie at least partially in their substantial – though not complete – ability to provide learners with multiple linked representations as described by Kaput (1992). TI-92 users may be able to move among various representations of the same functional relationship more flexibly than calculator and CAS users in earlier studies. The hand-held nature of the TI-92 (versus more public computer screens in prior CAS studies) could influence learning in contrary ways, enhancing the individual's constructions but stymieing social constructions. It seems then that the TI-92, which combines both the power of a CAS and the simplicity of a user-friendly graphics calculator, could have substantial impact on students' learnings during their first formal encounters with functions in a first-year algebra course. The current study addresses two related questions: What strategies involving the TI-92 hand-held CAS do first-year algebra students develop and use to explore real-world problems and to investigate properties of functions? What obstacles do these students encounter?

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Subjects and Context

The study investigates the strategies implemented by grade 9-11 students in a first-year algebra course at a large and culturally diverse urban high school. Their course used *Concepts in Algebra: A Technological Approach* (CIA) (Fey et al., 1995), a computer-intensive, concept-driven algebra curriculum that assumes continuous access to computing tools. Students used the calculator during class meetings to produce and manipulate graphs, tables and symbolic representations of functions as to fit curves. The course was taught by a secondary school mathematics teacher prepared to use TI-92s and related technology in CIA through three summer institutes.

Data Collection and Analysis

Individual interviews were done at each of three times during the 1995-1996 school year: when students first learned basic commands and used them with linear functions (December), after the students studied quadratic functions (March), and at the end of the school year (May). Subjects interviewed were all those who returned positive parental permission slips and whose schedules included an open period. Primary data include three individual interviews with each of five students. Each of six other students participated in one or two interviews. All interview tasks were developed and individual interviews were conducted by the author and a graduate student.

Each of the 25 interviews lasted 30 to 45 minutes, presented subjects with real-world settings and proposed function models (symbolic rules), and then asked subjects to answer mathematics questions about the situation. Two kinds of questions were of particular interest. A sample situation and input/output question appear in Figure 1. Output/input questions provide information about the input value and require students to determine the corresponding input value(s). A sample appears in Figure 2. Figure 3 shows various tool-based approaches that CIA students might use to answer the sample questions in Figures 1 and 2. After answering a question, a subject was asked to identify and use other strategies to answer the same question. After discussing these approaches, subjects were asked to compare methods and to react to alternatives.

The University of Nebraska football team won the national championship last year. As a reward, each member of the team received a ring that had diamonds in it. The cost in dollars of each ring depends upon its total weight in carats. The rule

$$d(w) = 1091\ w + 304$$

describes this relationship.

One ring has a total weight of four carats. What is the cost of this ring?

Figure 1. Hypothetical situation and input/output question from first interview.

Advertisements for Total cereal claim that Total has more nutritional value than other leading brands. The amount of protein in grams that you get from Total depends upon the number of servings that you eat given by the rule

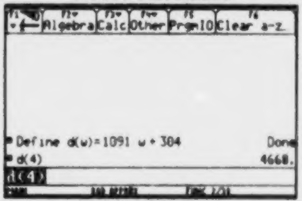
$$p(n) = 1.51\,n - 1.02\,.$$

In order to get 85 grams of protein, a recommended daily allowance, how many servings of Total should you eat?

Figure 2. Proposed situation model and output/input question from first interview.

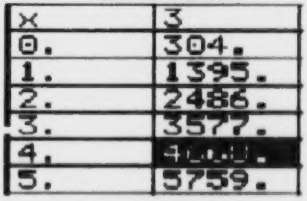
FOR INPUT/OUTPUT STRATEGIES

Direct Calculation



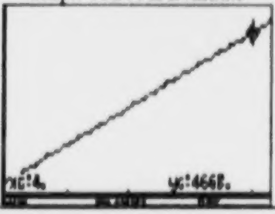
```
Define d(w)=1091 w + 304          Done
d(4)                              4668.
d(4)
```

Table of Values



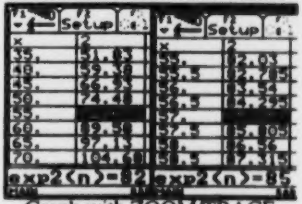
x	3
0.	304.
1.	1395.
2.	2486.
3.	3577.
4.	[illegible]
5.	5759.

Graph TRACE/Evaluate



FOR OUTPUT/INPUT STRATEGIES

Consecutive Tables of Values



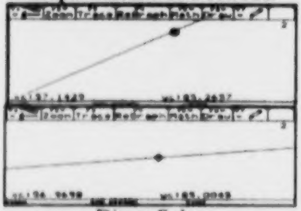
x	2
35.	51.83
40.	59.38
45.	66.93
50.	74.48
55.	
60.	89.58
65.	97.13
70.	104.68

exp2(n)=82

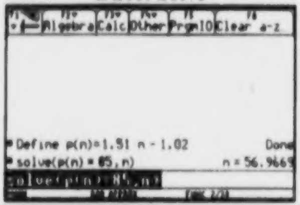
x	2
55.	82.03
55.5	82.785
56.	83.54
56.5	84.295
57.	
57.5	85.805
58.	86.56
58.5	87.315

exp2(n)=85

Graph with ZOOM/TRACE



Direct Solve



```
Define p(n)=1.51 n - 1.02         Done
solve(p(n) = 85, n)         n = 56.9669
solve(p(n)=85,n)
```

Figure 3. Expected CIA student strategies for questions in Figures 1 and 2.

To allow interviewers to see students' tool use, students used a TI-92 connected to a viewscreen with an overhead projector producing an image of the subject's screen on the wall behind the subject. Subjects regularly used similar equipment in the classroom. Interviews were audio taped and, pending parental consent, video taped when possible. Interviews were transcribed verbatim; transcripts were augmented with interviewer notes about subjects' original strategies, alternative strategies, and discussion of alternatives.

Three series of whole-class observations in the class of 23 students complemented interview data. The observations were done during the two to three consecutive days over which the interviews occurred. Audio tapes were supplemented by copies of students' written work, copies of teacher-generated curriculum materials, and a log of classroom

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activities kept by the teacher. Relevant portions of observations were transcribed verbatim. All data analysis followed Merriam (1988).

Results and Implications

Results address how the students used the CAS for symbolic, numeric, and graphic representations and then used these representations to answer questions about real-world situations. Following a brief description of general trends in subjects' strategies is an elaboration of how these strategies reflected technical and cognitive obstacles that students encountered.

Strategies. The subjects' uses of the tool fell into one of two categories: tablers and conquerers. All subjects except Henry were "tablers." For every task during all interviews, their initial approach used a table. The use of tables evolved over the year, as some subjects began using different starting values and increments, and paging through tables more quickly. The subjects rarely used graphs as a first approach. They seemed to avoid using graphs, noting that they could not determine a reasonable graphing window. When provided with the window, subjects used the TRACE or EVALUATE features of the TI-92. Few students used a direct computation, although several noted that the table was a collection of these computations. Subjects also avoided the direct solve command, claiming that its more complex syntax involved too much typing. As the year progressed, some but not all subjects rejected or revised strategies when two or more different approaches produced seemingly different results.

One student, Henry, was a "conquerer"; he used the tool very differently from other interviewees and classmates throughout the school year. Henry's first alternative for output/input problems used a direct solve. For input/output problems, he used a direct computation. When asked for alternatives, he demonstrated graph and table approaches. He alone seemed to have conquered all CIA strategies and to connect the approaches. For example, he explicitly explained how he used the result from the direct solve to determine the viewing window.

Obstacles. Subjects encountered obstacles that blended characteristics of the tool with evolving mathematical understandings. Several definite types of obstacles arose: viewing window determination, input/output confusion, parameter interpretation, and problem structure identification. *Viewing window determination* is the most clearly tool-related obstacle and it persisted throughout the school year. It refers to subjects' reluctance to use graphs because of professed or actual difficulty in establishing appropriate viewing windows. Subjects said that they expected to find the window values "in the question" (Carry, 3/8/96) or that "the teacher usually [figures out the window]" (Walt, 3/7/96). Only Henry consistently generated appropriate windows through all three interviews. However, classroom observations as well as interview situations indicated that subjects could be successful in using graphs to answer the questions once a reasonable window was established. Lack of success in using graphs or other representations was usually related to the second obstacle – input/output confusion.

For CIA students to be successful in using a tool-based approach, they must distinguish between as well as relate the input and output variables in the situation, in the function rule, in the table/graph/solve command/direct computation representation, and in the tool formulation of these representations. *Input/output confusion* in any one of these four arenas is an obstacle to successful problem solving. For example, Henry (12/15/95) first answered the Total Cereal question (Figure 2) using Solve(p(n)=85, n) to get 56.9669 and then computed 57x1.5+1.02 to explain why his answer of 57 servings made sense. When asked for an alternative approach, he used a graph in a window with n ranging from 0 to 57 and p(n) ranging from 0 to 85. He explained why he chose those numbers based on this answer from the direct solve approach. However, he then entered "85" for the "xc" (input value), got "ERROR: Window variables domain" and explained that 85 was out of the window. He then entered "84" for "xc" and got the ERROR again. Henry then went back to his window values and indicated with surprise that he needed to use "57" for "xc" and then TRACE to get the output near 85. Henry used the calculator with ease yet struggled here with the tool's formulation of input and output in the TRACE/ EVALUATION mode. His confusion over input and output in the tool's formulation of the graph was an obstacle. This example underscores the importance of understanding all four manifestations of the input/output relationship.

The third general obstacle was *parameter interpretation* within the function rule. In the first interview, for example, Carry interpreted the t(p)=55.37p+4.79 for amount of time it takes a computer to print a book as a function of the number of pages in the book saying "55.37 would be ... how many pages it did ... in like 4.79 seconds" (12/15/95). She and others talked about the input variable as a label and did not talk about its coefficient as a rate of change. Some subjects also generalized this interpretation scheme to quadratic function rules, giving each term a value-label meaning. This obstacle did not interfere with subjects generating answers as much as it caused them to doubt or reject answers when trying to explain how these answers made sense. For example, LaDonna said that "1091 pounds must be the weight and the cost is $304 per diamond" (12/15/95) when she read the Diamond Ring Situation. As a result, she said her (appropriate) answer of $4468 for the four-carat ring did not make sense.

A fourth obstacle to their success was *problem structure identification* and associating structures with particular tool actions. For example, Erika responded to the Total Cereal question by entering p(85). She rejected 127 servings as far too many and tried a table starting at 85, getting and rejecting 127 again. Erika seemed unaware of the inherent connection between the computation and the table and described their inappropriateness for the task based only on her real-world sense making. LaDonna (12/15/95) debated about whether "we have to solve" when she first read the situation in Figure 1 but abandoned the idea when she could not remember the syntax. Her first answer ($4468) came from a table that started at 4. She next got the same answer by entering d(4). Saying she really thought she should use Solve(d(w)=4,w), she entered this to get w = - .274877. She quickly said that she should not get the same answer using solve as she did for the table starting at 4 and the direct calculation of d(4). She decided that "on this particular problem" the table or direct computation was better because the cost should be "bigger than .274877" and positive. LaDonna stumbled on a valid answer because of her inability to recall syntax followed by her perception of reasonable values. Both LaDonna and Erika reflect the difficulty that subjects had in associating which tool-based activities were equivalent and appropriate approaches for either an output/input task or an input/output task.

Conclusion

The results suggest that students develop different strategies yet encounter obstacles when using the TI-92 in solving real-world problems. These strategies seem to reflect a complex interaction of students' understandings of fundamental mathematical issues and students' perceptions of the tool's integrated features. Deeper insights of how understanding of mathematics and tool co-develop suggest a delicate balance between conceptions and keystrokes in the classroom.

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DIAGRAMMATIC REASONING, TECHNOLOGY AND GEOMETRIZATION OF PROOF IN SECONDARY MATHEMATICS

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Mathematics education research suggests that computer-based imagery has a potential to play an important role in developing mathematical thinking and in acquiring abstract concepts by the students (Sutherland & Mason, 1995). One abstract concept that has been a key element of mathematical practice from antiquity to present time is that of formal proof. How can the power of technology be used to promote the idea of proof in secondary mathematics? How can this power be put in work to make mathematical proof understandable by highlighting its major arguments? The presentation deals with the geometrization of inductive and deductive proofs within a computational medium; that is, it suggests using computer-generated diagrams, numerical patterns and graphs as a cognitive support for comprehending underlying logical structures and accompanied algebraic symbolism of the proofs.

The author argues for an important role of a computer as a mediator of one's articulated presentation of proof in a symbolic language of mathematics rather than a medium that one utilizes to replace proof by measuring, computing, or graphing. Consider a proof by mathematical induction — a most common method of proof in secondary mathematics. According to Pólya (1954) two phases are involved in mathematical induction proof: inductive and demonstrative. In a technology-rich environment both inductive and demonstrative phases can be supported visually; the former—by computer generated numerical evidence, the latter—by diagrams and graphs. Furthermore, the numerical evidence has a potential to mediate students' learning conscious awareness and mastery of a proof by mathematical induction.

Computer-based diagrammatic reasoning can be also used effectively as a support system in proving mathematical results through deduction. The presentation, based on work done by the author with preservice and inservice secondary school teachers, shows how diagrammatic reasoning mediates the transition from numerical evidence to algebraic symbolism of deductive proof through its geometrization.

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ENACTIVE ENVIRONMENTS FOR PROBLEM SOLVING: MANIPULATIVES VERSUS EQUATIONS

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Mathematical problem solving beyond the elementary level, especially solving word problems, is usually based on the use of techniques and strategies involving the transition from natural language sentences to the symbolism of variables and equations with a subsequent finding of the solutions in a numerical or symbolic form. Yet, successful strategies in algebra can integrate cognitive and physical actions on concrete embodiments in order to obtain solutions. Using electronic manipulatives with a particular emphasis on a spreadsheet-oriented semiotic mediation in modeling word problems demonstrated the effectiveness of the approach in teacher education courses (Abramovich & Nabors, 1996).

The presentation further develops the authors work and shows how a spreadsheet was used by seventh grade students as an enactive environment for problem solving. It was found that reasoning in a trial and error form allows learners to use grouping strategies structured by quantitative relationships related to a particular problem situation. These activities lead to an iconic representation of a solution from which its numerical form can be drawn. Thus we argue that variables and equations are not the only means of obtaining a solution.

The students were presented with tasks that led them from initially becoming acquainted with a spreadsheet, to manipulative and numeric approaches to a variety of problem situations, and finally, to the manipulative, numerical and generalized solutions of non-linear problems, in particular, to work problems.

Based on observations, the authors suggest that the spreadsheet environment provides the setting for supporting critical aspects of mathematical thinking; namely, the use of cell-patterned diagrams and iconic representations of mathematical relationships mediates algebraic reasoning through performing cognitive and physical actions on them; the combination of manipulative and numeric approaches enables students to generate new meanings through explorations based on visualization and inductive reasoning; and the activities support conjecturing that goes beyond an original problem, thus indicating the move from problem solving to problem posing.

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A CAS LEARNING STUDY: THE IMPACT OF ACCESS TO A COMPUTER ALGEBRA CALCULATOR ON THE USE OF MULTIPLE REPRESENTATIONS BY PROSPECTIVE MATHEMATICS TEACHERS

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Previous research on multiple representations using graphics calculators limited types of representations to numeric and graphical. Hand-held computer algebra systems (CAS) like the Texas Instruments TI-92 now offer accessible interactive symbolic, numeric, and graphical representations, with easy entries from geometric and concrete representations through a dynamic geometry tool and a calculator based lab tool (e.g., the TI CBL). Our research[1] focused on ways in which prospective teachers, equipped with these hand-held computer algebra systems, use multiple representations when learning mathematics related to their teaching.

Subjects are 17 prospective secondary mathematics teachers with at least 18 mathematics credits at the Calculus level and beyond, enrolled in a mathematics education course focusing on technology and mathematics learning and teaching. Students were loaned a TI-92 for in-class and out-of-class use. The class met for 15 three-hour sessions.

Major sources of data were audio and video tapes of classes and task-based interviews involving eight targeted students representing a range of mathematics achievement (mathematics GPA) and exploration tendency (performance on a nonroutine mathematics problem completed at the beginning of semester). Four to six observers constructed verbatim transcripts of the actions and interactions of target students while working alone and within groups. An informal discussion of the class and transcript construction immediately followed each class meeting. Hour-long nterviews were conducted four times during the semester. Each task-based interview includes two non-routine problems for which solutions by different representations and using TI-92s were likely. Verbatim transcripts of interviews were annotated with calculator screen content. Secondary data included student journals, examinations, and written assignments.

The project team identified and characterized use of multiple representations across problems as well as across individuals. Emerging results center on the following: breadth of approaches that students with similar mathematics backgrounds take when conducting personal mathematical explorations; relationships between familiarity with the tool and the nature of student explorations; the interplay of mathematical understanding and tool use; and the relationship of tool use and the flexibility of the problem-solving approach.

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AN EXPERIENCE WITH TEACHERS ABOUT THE CONCEPTION OF FUNCTION AND USE OF CALCULATORS WITH GRAPHIC CAPACITY

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In this document we briefly describe the work carried out in the Technological of Superior Studies of Ecatepec, in an activity developed in an investigation project in Mathematical Education that we are working on.

In this experience, we worked with five different mathematics teachers at the college level and we used a Texas Instruments TI-81 calculator. The purpose of the investigation consisted in the exploration and analysis of the following three criteria which were the basis for the observations, data, and results obtained during the development of the investigation:

- The conceptions of the teachers about the use of technological tools in mathematics teaching;
- The "new" conceptions of the mathematical concepts that have resulted from this use;
- The strategies and new options for solving mathematical problems.

We designed a series of problems that would allow us to explore with the calculators in respect to the criteria mentioned. We dedicated three sessions of three hours each to carry out the work. The first two sessions were designed to explore and solve the problems and the third for a scientific discussion [1] between the teachers. The analysis of the observations and obtained data was carried out by a study of qualitative character, trying to analyze the algebraic, numerical and graphical aspects involved [2]. The type of problems worked were as the following examples:

1. Find the value of "a" so that $f(a) = 2$ in the next functions: (i) $f(x) = x+5$; (ii) $f(x) = 1/(x+4)+2$; (iii) $f(x)=\sqrt{x^2-3}$
2. Solve $\cos x = \ln x$.

An important aspect here is that in the used strategies, the notion of function plays an important role in its graphic representation. Nevertheless, the approach of the teachers when solving an equation is directly associated with their beliefs about mathematics. For example, they frequently relied on algebra (isolating "x") to verify the results that appeared on the calculator's display.

In general the strategies of the solution to this and other problems and the opinions of the teachers (in the discussion) reflect important cognitive aspects. For example, the answers on the calculator are derived as a result of operating with "complete" objects, both an analytical and a graphical expression. Not just one process is explicitly considered. The operations of evaluation are substituted by the use of "trace" to determine some "x" of interest. These and other similar aspects, favor a conceptualization of function related with the creation of a concept image of operation with (graphic) objects with *general properties* associated with analytical expressions.

We think it is important to continue working on the didactic and cognitive elements involved in the use of this technology in teaching; it is also convenient to explore all the kinds

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of functions that commonly appear on text books, because calculators (at least the ones used by us) do not correctly graph some functions and this could cause conflicts if one is not careful and tries to introduce these technological aides in the teaching of mathematics.

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VISUALIZATION

USING COGNITIVE MODELS TO DEVELOP A VISUALIZATION ENVIRONMENT

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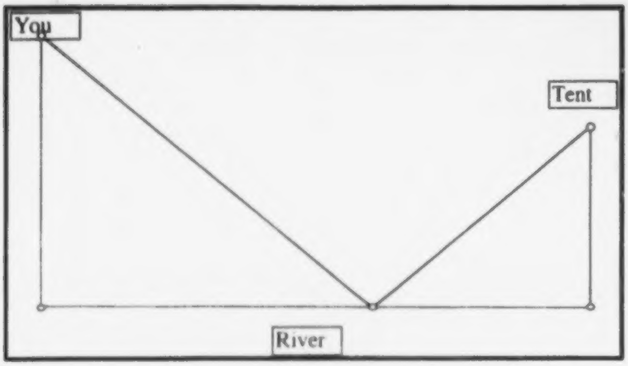
There has been little discussion linking the internal visual constructs created during interaction in a visual environment to cognitive theories in general. It is important to begin a theoretical framework for understanding the concepts of visualization and visual reasoning on which we can build visual environments for the purpose of research. Constructivism and information processing can help explain mathematics visualization and visual reasoning and form the foundations for such construction. If students' and teachers' use of visual reasoning are important, then research efforts should be directed toward describing and explaining these reasonings. This paper serves to draw together an idea of visualization through constructivism and information processing, use this idea to create a visualization environment, and offer implications for research.

Mathematics visualization has become a hot topic of discussion among mathematics educators. With the ever-increasing wealth of visualization technologies, creating an environment that is rich in visualization possibilities is easier than ever before. As mathematics educators, we may often put the students in these environments with little consideration of their thought processes. There has been little discussion linking the internal visual constructs created during interaction in a visual environment to cognitive theories in general. It is important to begin a theoretical framework for understanding the concepts of mathematical visualization and visual reasoning. I will show how constructivism and information processing explain mathematics visualization and visual reasoning and form the foundations for constructing a research agenda.

Mathematical visualization and visual reasoning have undergone somewhat of a renaissance with the widespread accessibility of computers and graphical programs. Computer graphics have greatly expanded the scope and power of visualization in every field. The term "visual reasoning" is unfamiliar to some in the context of mathematics, and its connotations may not be obvious. In visual reasoning, we are interested in the student's ability to construct an appropriate external representation of a mathematical concept or problem and to use the representation to achieve meaning, and as an aid in problem solving (Cunningham, 1991). In mathematics, visualization and visual reasoning are not ends in themselves, but a means to an end, which is meaningful mathematics learning. Visualization is the process of forming images and using these images effectively for mathematical discovery and understanding.

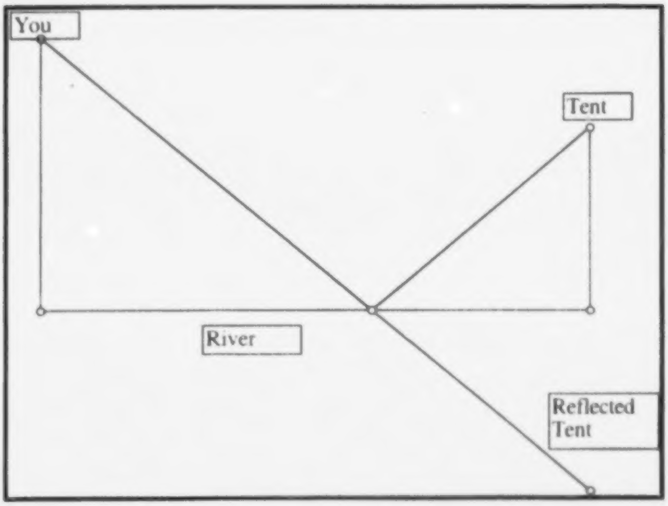
For example: you are 3 miles from the bank of a straight river. Your tent is 2 miles from a point 6 miles downstream. You must stop for water on the way to the tent. Where must you stop to minimize your trip? The visualization below helps depict the relation ship of you, the tent, and the river and helps make sense of the problem situation.

Figure 1



Visual reasoning is present in a solution that reflects the tent across the river and concludes that a straight line represents minimal distance.

Figure 2



To understand what constructivism has to say about visualization, we must first understand constructivism as an epistemology. Constructivism relies heavily on the idea that cognition serves the student's organization of the experiential world (von Glasersfeld, 1990). Reflective abstraction, repeated experience, and knowledge hierarchies make up the basic moving parts of the mechanism of constructivism. Reflective abstraction is the act of abstracting commonalities and rules from experiences. Repeated experience consists of similar situations being experienced by the cognizing individual. Since no two situations are ever completely alike, regularity is achieved by disregarding some of the differences. If cognition is adaptive, tending towards fit and viability as constructivism suggest, then as educators, we must prepare experiences that show several different aspects of a topic. In the visualization classroom, this means several different media used to represent the same mathematical concept.

Anderson (1983) characterizes knowledge in two forms, declarative and procedural. Further refinement of these, by the way the information is encoded, suggests consideration of verbal and non-verbal systems (Paivio, 1986). Greeno (1989) characterizes the interplay of these systems as he connects symbolic and manipulative perspectives. Thus visualization and visual reasoning can be considered from a manipulative perspective and concepts can be defined. Consider a concept as a cluster of declarative and procedural knowledge that is formed from the symbolic and manipulative representations of a given topic. The act of encoding and decoding a visual representation from interaction with manipulative environments takes place as the creation of a concept. Information processing deals directly with interpretation in visualization through schemas and concept images.

The formation of mathematical concepts occurs via two steps, abstraction and classification (Skemp, 1987). Abstraction is the act of determining the crucial similarities among a group of objects. Classification uses the similarities to determine if a new object belongs with the aforementioned group. The abstraction that has been created is called a concept. Skemp calls the structure that connects these concepts a schema. The mental construct that is developed by a student to understand a concept can be referred to as a concept image. It consists of all the schemas as well as "all the mental pictures and associated properties and processes" of a given concept (Tall & Vinner, 1981, p.160). The introduction of graphs, animations, and other visual representations is of the utmost importance because this is visualization's contribution to the concept image.

From the preceding arguments, five ideas for creating a visualization environment become evident. First, as teachers, we must allow time for reflective abstraction. We must give the students several attempts at relatively similar problem situations, thus providing repeated experiences. We must also monitor students as they create knowledge hierarchies. Also, we must provide logical and connected visual representations. Finally, we can talk about visualization as a manipulative environment that expands the symbolic perspective.

With these five ideas in mind, i constructed a visualization environment to teach trigonometric functions utilizing several technologies. The unit is composed of hypertext markup language (html) documents or "pages". Interactive exercises within the unit were prepared using the *Geometer's SketchPad* (GSP)(Jackiw, 1992) and spreadsheets.

First the students encounter an overview of the unit. Successive pages take the students through the construction of the trigonometric lengths on gsp, the geometry inherent in the trigonometry, the graphs of the trigonometric functions, and the algebraic manipulations of these graphs. At each level, students are given the links to software packages and encouraged to explore.

Experiences are repeated through the six trigonometric functions. The choice of topic allows students to study the similarities within pairs of trigonometric functions as well as global similarities of all the functions to allow reflective abstraction. The close connection to gsp and its inherent abilities allows students to manipulate and animate many of their constructions. This creates a knowledge hierarchy of motion.

Finally, considering visualization as a manipulative environment was a factor in choosing the area of trigonometric functions. Many teachers may become frustrated teaching the material from a symbolic perspective. Using the visualization environment allows students to get a different idea of trigonometric ratios. They can group the information they derive from the gsp environment with the symbolic perspective they already had. Students exposed to the visualization environment have an imaginal system, and talking about the pictures gives them a verbal system at the same time.

These perspectives are useful in forming a bigger picture of visualization and the mathematics student's mind as well as creating visualization environments. To follow a constructivist view, we must prepare lessons built upon students' prior experiences and classroom experiences, thus allowing them to construct their own meanings. We must prepare experiences that point out several different aspects of a given topic to aid in student's reflective abstraction as well as giving repeated experiences with the same information. At the same time we are reminded that no two children will create the same knowledge from the same visual experiences. We must, therefore, make an attempt to understand the knowledge hierarchies that the students are forming as a result of their experiences in the visualization environment.

Information processing tells us that in order to cluster declarative and procedural knowledge from the different symbolic and manipulative perspectives, we must provide logical and connected visualization experiences. It is precisely the ability of the visualization environment to provide a manipulative perspective that makes visualization such a vibrant area for future growth. Perhaps this glimpse has shown some of what lies beneath the surface of a visualization experience and has given some insight into possibilities for study of these experiences.

Implications For The Future

If students' and teachers' use of visual reasoning are important, then research efforts should be directed toward describing and explaining these reasonings. The specific research questions to be addressed will follow from making theories explicit and predictions testable. Hopefully, i have begun to draw together an idea of visualization and visual reasoning through constructivism and information processing. The visualization environment seems fruitful for research. However, these ideas may completely determine neither the research design nor other particulars of the research process without further refinement and discussion.

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THE INFLUENCE OF VISUAL PERCEPTORS IN INTERPRETING THREE DIMENSIONAL DIAGRAMS

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The purpose of this study was to explore the nature of visual perception used by students of mathematics in South African schools. Participants between the ages of 12 and 18 were interviewed using 3-D diagrams prevalent in the mathematics classroom. Diverse perceptions reported by participants suggest that representational cues do not dominate students' perceptual activity. The concept of visual perceptors is introduced as a useful alternative to the traditional role of representational cues in influencing students' perceptions of 3-d diagrams. Four visual perceptors used by students are suggested and evidenced.

In the learning of mathematics students are at some point expected to give meaning to diagrams which could be interpreted as three dimensional. Such diagrams we refer to as 3-D diagrams, and define as two dimensional representations of three dimensional space based on the conventions of society. However, there is often an assumption made by mathematics educators that students have the necessary visualisation ability and experience with the conventions of 3-D diagrams to appropriately interpret diagrams presented to them in the classroom (Bishop, 1986). The purpose of this paper is to examine the nature of students' visual perception of 3-D diagrams as it relates to the leaning of mathematics.

Research Methods

South African high school students were asked to interpret and describe their perceptions of 3-D diagrams. Written responses as well as verbal self-reporting were used as the primary sources of information.

Participants

Four groups of South African high school students participated in the initial testing. Three students from each of the four groups were then selected to participate in semistructured, task based interviews which were audio-recorded. Ethnic and gender diversity were encouraged. Both black and white students, boys and girls were interviewed. The inclusion of diverse groups is an attempt to gain as broad an understanding of south african students' perceptual activity and visualisation as possible and not to create a comparative study. Students, aged 12-14, beginning their eighth as well as students, aged 15-18, completing their tenth year of schooling were asked to participate in the research. These students were of average mathematical ability, as determined by school grades.

Initial Testing and Interviews

An initial test was given to each group of students in a mathematical setting. The initial test, which required the students to answer specific questions in written format, consisted of 3-D diagrams in parallel perspective (the perspective most commonly used in south african mathematics textbooks). The tasks included perceptual tasks as well as spatial visualisation and orientation tasks (McGee, 1979).the test acted as the basis for the selection of interviewees as well as providing insight into the students spatial ability.

The purpose of the interview was to elaborate on how the students came to perceive in the way they did rather than an elaboration of what the students perceived. Interviews were initiated by recalling students' written responses. Students were asked to elaborate on their perceptions as well as describe their solution activities. Translation facilities were available where necessary.

Visual Perceptors Used by Mathematics Students

A mutual understanding, between representor and interpreter, of the spatial conventions used in a 3-D diagram is necessary (but not sufficient) for effective visual communication in the mathematics classroom (Parzysz, 1988). Authors in the field of spatial perception suggest that the perception of the student is influenced by the "cues" contained in the diagram as well as previous experience with the conventions of the representor (Deregowski, 1980). However, even students who are familiar with the conventions may not perceive the diagram in the way intended by the representor (Parzysz, 1991). This suggests that the interpreter's perception may be significantly more influential than some authors (e.g. Deregowski, 1980) previously claimed.

Berkeley claimed that, "we identify them (pictures) and act not so much according to what is directly sensed but to what is believed" (Gregory, 1970, p.11). The student's "believed" perception of a 3-d diagram is determined by a mental construction of the image rather than the properties of the diagram. The diagrams are perceived through the "lenses of our past", making spatial perception as much a cultural phenomenon as mathematics (Gregory, 1970; Bishop, 1989). Consequently the perception of a diagram is influenced by both psychological and sociocultural factors associated with the interpreter (Cohen & Akarsu, 1991; Cobb & Yackel, 1995).

Our data illustrate that 3-d diagrams experienced by a student in the mathematics classroom are not perceived on the basis of spatial conventions contained in the diagram but rather on the basis of **visual perceptors** used by the student in the interpretation of the diagram. Visual perceptors are a component of the mental visualisation process of the interpreter which influences how children perceive conventions used in a diagram. Students invoke visual perceptors on the basis of past experiences, both with physical objects as well as diagrams.

We argue that the children's perception is primarily influenced by perceptors and not the cues contained in the diagram. Therefore the role we attribute to these perceptors replaces the role of the `visual cue' used by other authors (deregowski, 1980). A suitably drawn diagram may facilitate the use of an appropriate visual perceptor, however the use of a visual perceptor cannot be determined by the diagram.

Four visual perceptors, namely, spatial perceptors, environmental perceptors, geometrical (or mathematical) perceptors and artistic perceptors have been identified during our analysis of the interview protocols. The use of a particular visual perceptor does not appear to be persistent and thus given the same diagram at a later time the perceived image may well vary.

Spatial Perceptors

A spatial perceptor is the use of the knowledge of a spatial convention in the interpretation of a diagram. A knowledge of the spatial conventions used by the representor is necessary for the effective use of a spatial perceptor. A spatial perceptor results in the students focusing their attention on a specific spatial convention in the diagram and interpreting the diagram on the basis of their experience with the convention.

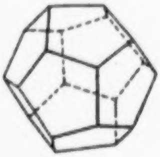

Figure 1. How Many Sides Does This Object Have?

101 D: There is 1, 2, 3, 4, 5 and then 6, and a 7, 8, 9, 10, 11, 12. (counted shapes enclosed by solid lines as well as broken lines)
102 I: Right. Why do you count these dotted lines?
103 D: Because i know the other sides.
109 I: have you seen dotted lines before?
110 D: Yes
111 I: What does a dotted line usually mean to you?
112 D: To show the other lines that you can't see.

Delon focused his attention on the dotted lines (line 101) resulting in his counting 12 faces rather than counting six faces and then multiplying by two. Delon's use of a spatial perceptor is illustrated by his use of the dotted lines (line 112) in interpreting figure one. Delon's use of a spatial perceptor has focused his attention on the spatial conventions used in the diagram. The value of using a spatial perceptor is determined by the interpreter's knowledge of the conventions of diagrammatic representations.

Environmental Perceptors

An environmental perceptor results in the interpretation of the diagram as a representation of some experienced object. The protocol illustrates how figure one is described, by the student, as a previously experienced object. The attention to spatial conventions is replaced by a focusing on the environmental aspects of the diagram. Thus the diagram is perceived as a previously experienced object rather than a mathematical representation of space.

159 I: And this one here, how do you get twelve?
160 R: I said 1, 2, 3, 4, 5, 6. I said six then it's like, then let me make an example with a soccer ball. I turn it around to see what's on the other side and i see the same blocks. Say it's timesed 2 it gives me 12.

Raymond established a three dimensional interpretation of figure one by means of an environmental perceptor, based on his experience of a soccer ball (line 160). His visual image of a soccer ball along with arithmetic manipulation allows him to successfully determine the number of sides illustrated in the diagram (line 160). The environmental perceptor used by raymond allowed him a useful means of appropriately manipulating the diagram. It appears that in the south african context environmental perceptors are the most frequently invoked means of perceiving and interpreting three-dimensional mathematical diagrams.

Geometric Perceptors

A geometric perceptor results in the student perceiving the diagram as a mathematical illustration and consequently ascribing a geometrical label to the diagram. This is displayed by the use of geometric nouns or adjectives in the verbal description of the student's interpretation of the diagram. We suggest that the use of a geometric perceptor is most frequently related to the student's experience of formal geometries. The use of geometric perceptors was substantially less commonplace than the use of both environmental or spatial perceptors.

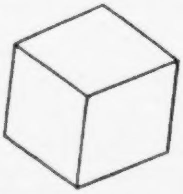

Figure 2. What do you see in the picture?

209 I: In this picture here what do you see?
210 W: A square
211 I: A square. Just one square?
212 W: No
213 I: How many?
214 W: Three

Wellington displayed the use of a geometric perceptor in his interpretation of figure two by the use of the term "square" (line 210). Square is used as a noun rather than an adjective illustrating that he perceives it as a square rather than as a square shape, which is consistent with the fact that no geometrical square shapes are present. Wellington used his knowledge of the geometric nature of a square to make sense of the diagram presented to him.

Artistic Perceptors

An artistic perceptor is the use of an artistic perspective in the interpretation of the diagram. The use of an artistic perceptor results in the attention of the student being focused on the aesthetic appearance of the diagram. Artistic perceptors were not prevalent in the research, a possible consequence of the testing environment.

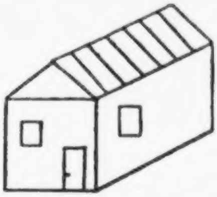

Figure 3. Circle any mistake in the picture (If you think there is a mistake).

149 I: Why did you say that's wrong?
150 H: Because this one is a little bit shorter than that door and you can't see out of the windows.
151 I: All right, so then, what's wrong with these windows?
152 H: They're too high.
153 I: Okay and is there anything else wrong with that picture?
154 H: Not that I can see.

Heidi focuses her attention on the artistic nature of figure three rather than the breach of spatial convention. Her use of an artistic perceptor is illustrated by her concern that the windows have been drawn too high for people to look out of them (line 150). Her ability to interpret spatial conventions is displayed on many occasions during the interview. We suggest that she is not unable to see the error (line 154) but rather that she ignores the window on the side of the house that has been drawn out of perspective because of her use of an artistic perceptor rather than a spatial perceptor.

Usefulness for Mathematics Educators

The identification of visual perceptors, used by students in their perception of diagrams, provides a tool for mathematics educators in their efforts to understand the nature of students' visual perception. The use of a particular visual perceptor influences the students' perception which in turn influences how they respond to the diagrams presented to them. An awareness of students' perceptions may foster improved communication in the classroom. Students may be encouraged to use their most effective visual perceptor thereby alleviating some of the spatial perception difficulties they encounter. This was illustrated by the effective use of environmental perceptors in interpreting 3-D diagrams.

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RELATING MENTAL IMAGE AND LANGUAGE USE TO CHANGE IN TEACHER BELIEFS

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The route toward change in beliefs involves more than a cursory assent to a presentation of new ideas or practices. The dynamics of belief systems involve various strengths of psychological commitment and multiple modes of clustering (Green, 1971). While recent research in mathematics education on teachers' beliefs and conceptions has focused on change, further investigation is needed into how the process of change proceeds from knowledge to belief and practice. The research in this presentation explored the possibilities of challenging inservice teachers' beliefs through a double-faceted reflective dialogue staff development program that included classroom videotaping, individual and group reflective analysis and discussion.

The subjects in this one-year study were three 4th and 5th grade teachers who volunteered to work with the researcher. This experience presented the teachers with the opportunity to revisit their teaching, acknowledge the rationale and beliefs underlying their practice, discuss and debate issues with their peers and move toward change. The study, utilizing both quantitative and qualitative methods, analyzed the data for patterns and trends. Through grounded theory methods, categories of teachers' use of language and mental representations of mathematics teaching emerged. These were further analyzed by teacher and across teachers in relation to their changing practices and beliefs to determine the usefulness of the categories. A construct developed that modeled the movement of teachers toward change. Their newly acquired knowledge of reform led to either immediate rejection or a change in their mental image of teaching mathematics. A change in mental image led back to a quest for more knowledge or to experimentation in classroom practices that aligned with the new image and/or experimentation at the verbal level to discuss the ideas further and to gain acceptance and confidence from peers. All these facets preceded and scaffolded a change in belief. Following belief change, teachers were more confident in instituting change in practice and were more fluent and comfortable in their discussions of reform in mathematics teaching and learning.

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THE DEVELOPMENT OF ABSTRACT THOUGHT: THE ROLE OF MENTAL IMAGERY

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Understanding the role of imagery in the teaching and learning of mathematics is of particular importance now since the use of various technologies such as calculators and computers make the graphic representation of even complex mathematical problems possible. Graphical modes of presenting and analyzing data are quite prolific, most recently spawning the field of scientific visualization. The need for visualizing problem information is common to both student and scientist. In this paper we will briefly summarize some reasons why the visualization of information is so critical to the development of abstract thought. We will center our discussion on the implications of mental imagery for student learning.

Learning mathematics requires individuals to be able to engage in progressively abstract thought. Several researchers in mathematics education (Cifarelli, 1988; Goodson-Espy, 1994; Presmeg, 1992; Sfard & Linchevski, 1994; & Wheatley, 1992) have investigated the kinds of physical and mental representations that individuals use when they are involved in solving mathematics problems. This research has revealed that: (1) novices and experts have different expectations regarding the use of imagery; (2) inconsistent use of imagery by an individual is connected to his or her familiarity with the problem; (3) imagery may be used by an individual to advance the level of abstract thought they are capable of attaining; and (4) the use of imagery increases the potential for attaining more abstract thought levels but advances are not automatic. However, individuals who engage in mental imagery tend to be very flexible in their modes of analysis—an attribute very conducive to the development of abstract thought. Supporting examples will be drawn from a study (Goodson-Espy, 1994) that examined the problem solving activities of college students engaged in solving word problems involving the concept of linear inequality.

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